ARCHI 2017, Nancy, France — March 6–10, 2017

Hardware Implementation of Cryptography

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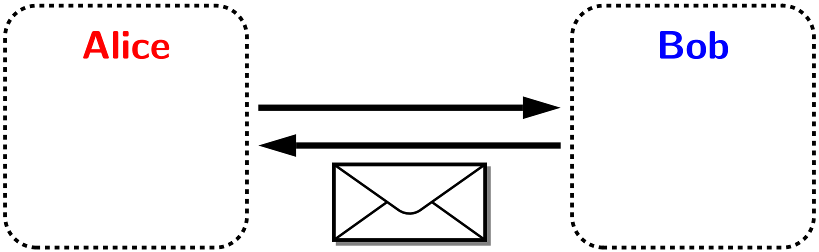

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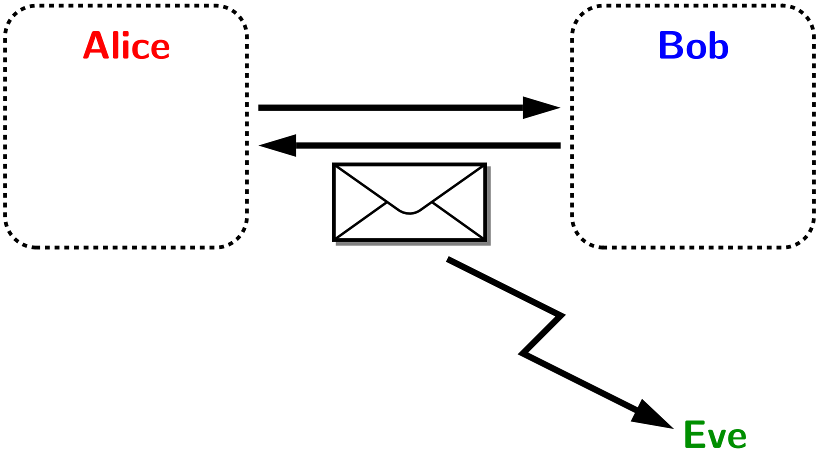

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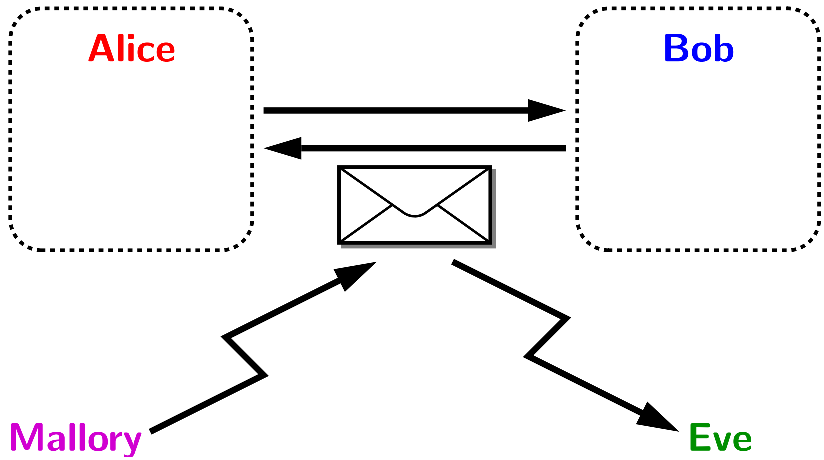

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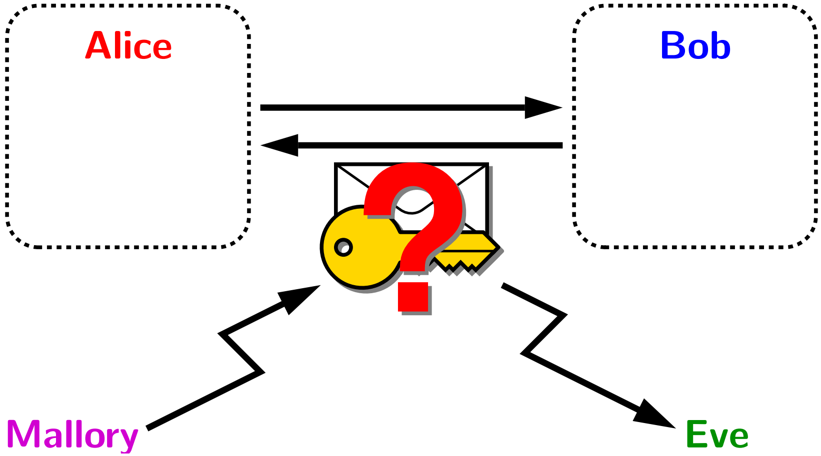

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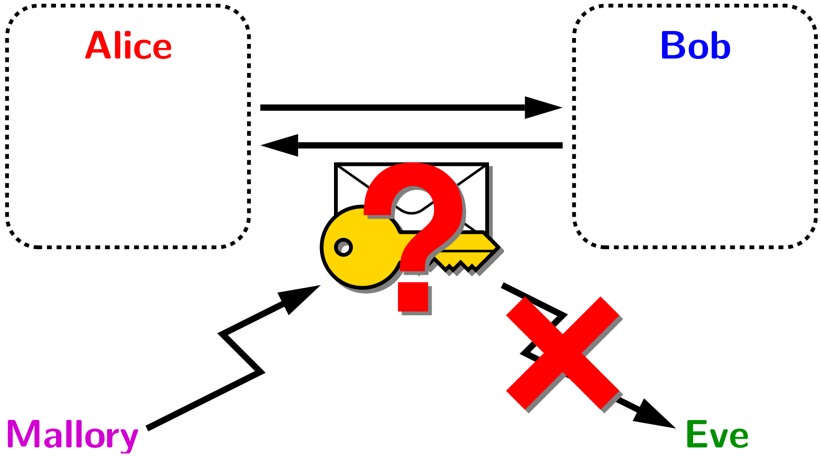

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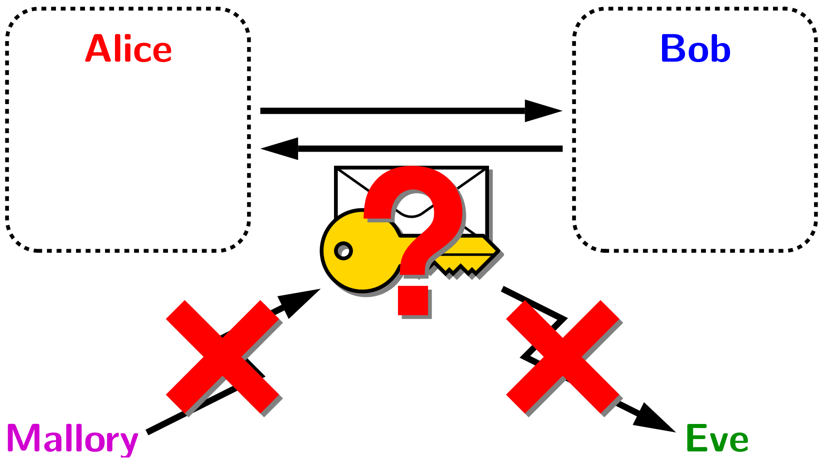

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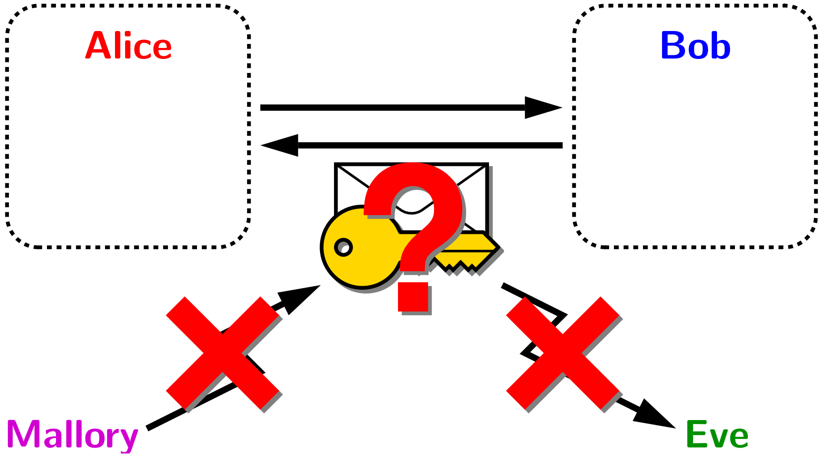

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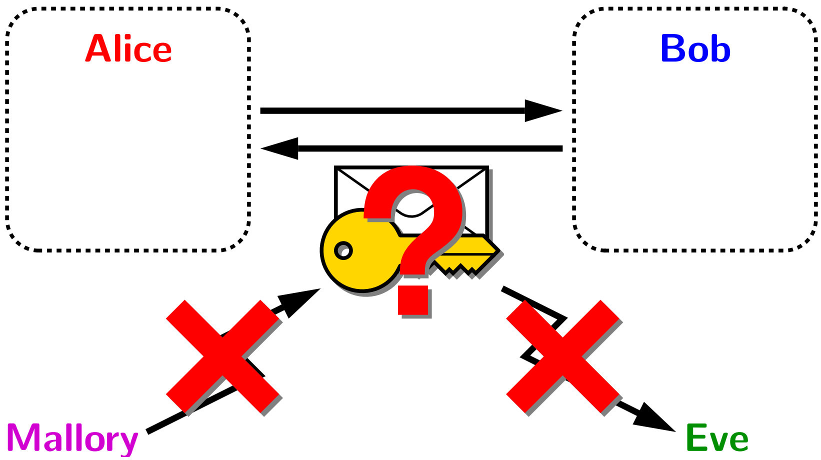

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  - ... and many others: non-repudiation, zero-knowledge proof, secret sharing, etc.

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- In this lecture, we will mostly focus on the **green layers**

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  $\Rightarrow$ In such cases, implementation security is usually less critical

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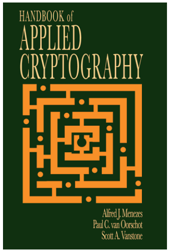
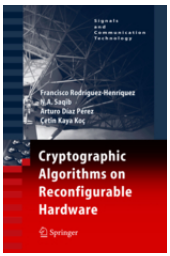
  - etc.

  ⇒ Possible attack scenarios depend on the application

Some references



Alfred Menezes, Paul van Oorschot, and Scott Vanstone,
Handbook of Applied Cryptography.
Chapman & Hall / CRC, 1996.
`http://www.cacr.math.uwaterloo.ca/hac/`



Francisco Rodríguez-Henríquez, Arturo Díaz Pérez, Nazar Abbas Saqib, and Çetin Kaya Koç,
Cryptographic Algorithms on Reconfigurable Hardware.
Springer, 2006.

Proceedings of the CHES workshop and of other crypto conferences.

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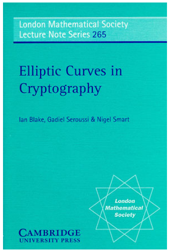
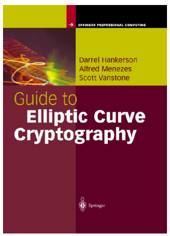
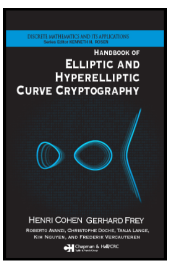


Elliptic Curves in Cryptography,
Ian F. Blake, Gadiel Seroussi, and Nigel P. Smart.
London Mathematical Society 265,
Cambridge University Press, 1999.



Guide to Elliptic Curve Cryptography,
Darrel Hankerson, Alfred Menezes, and Scott Vanstone.
Springer, 2004.



Handbook of Elliptic and Hyperelliptic Curve Cryptography,
Henri Cohen and Gerhard Frey (editors).
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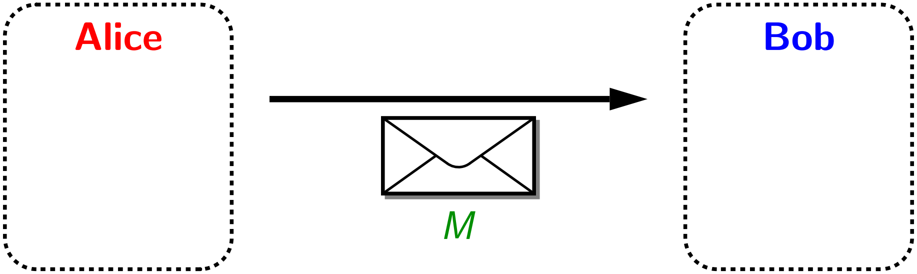
Outline

- Some encryption mechanisms
- Elliptic curve cryptography
- Scalar multiplication
- Elliptic curve arithmetic
- Finite field arithmetic

Symmetric encryption



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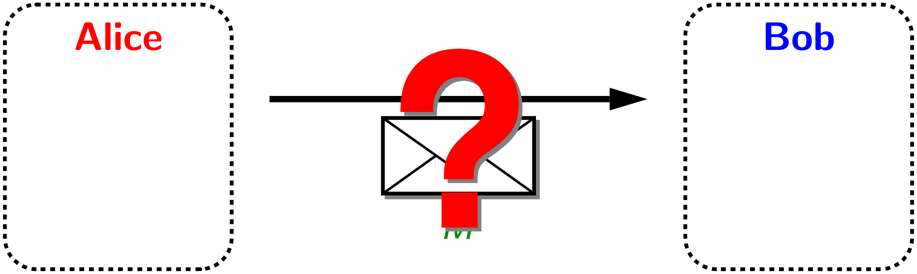

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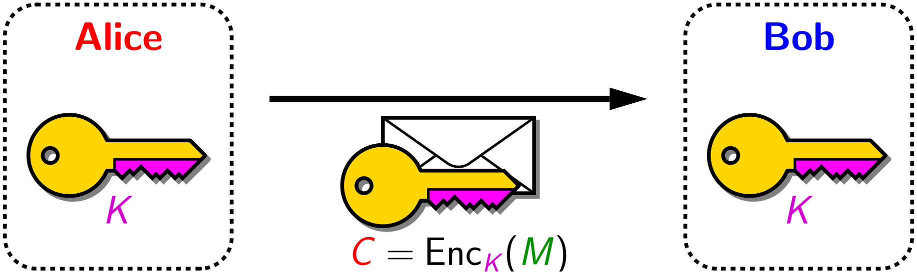

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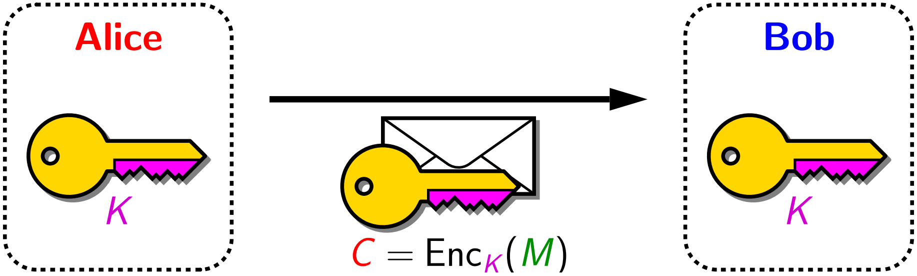

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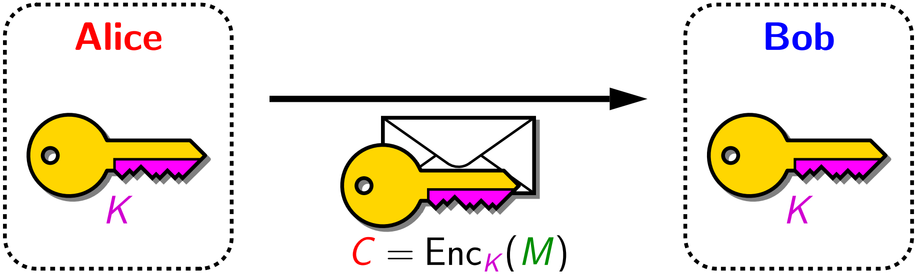

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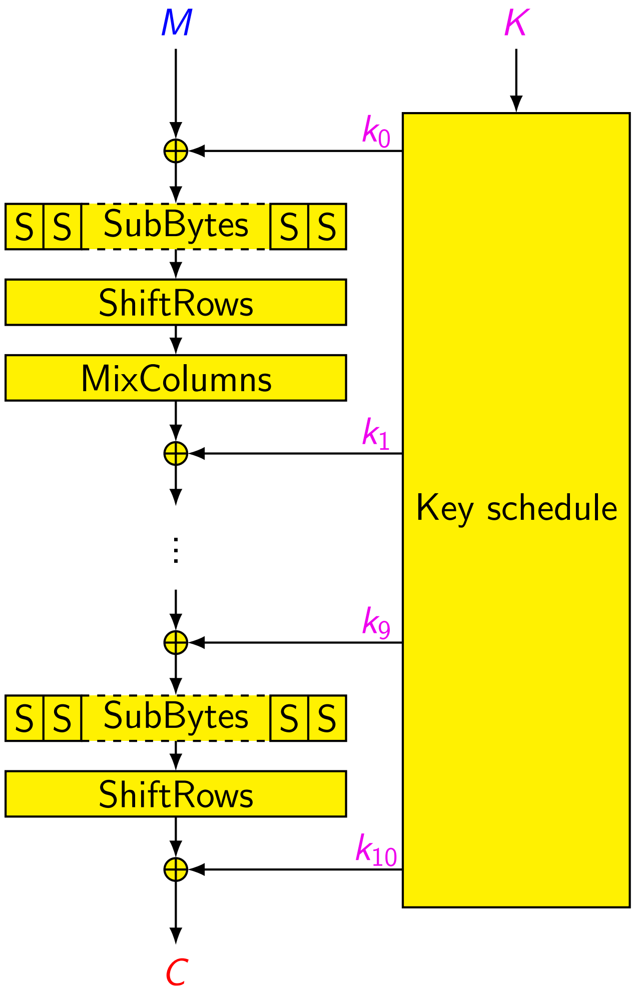
- Block cipher:
  - split message M into n-bit blocks (e.g., $n = 128$ bits)
  - encryption/decryption primitive : iterated keyed permutation $\{0,1\}^n \rightarrow \{0,1\}^n$
  - requires a mode of operation to combine the blocks

AES [Daemen & Rijmen, 2001]



- Advanced Encryption Standard
- Key sizes: 128, 192 or 256 bits
- Block size: 128 bits
- Substitution–permutation network
  - SubBytes: nonlinear subst. on bytes
  - ShiftRows & MixColumns: mainly wires, plus a few XORs
- 10, 12, or 14 rounds (depending on key size)

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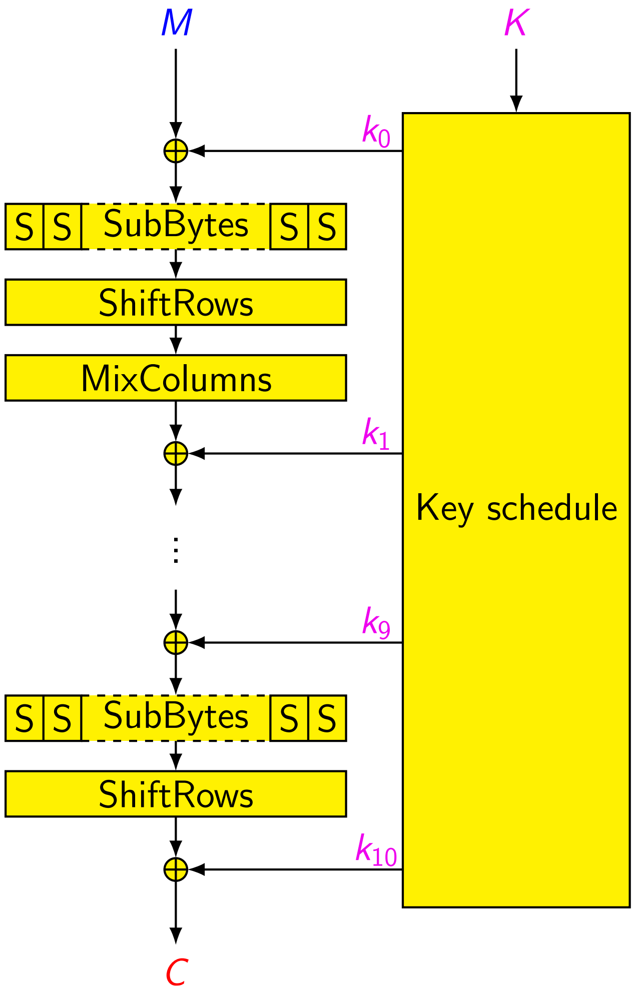

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- Low-area version (1 S-box): 20 cycles / round, 2.5 to 5 kGE
- Parallel version (20 S-boxes): 1 cycle / round, 20 to 35 kGE
- Fully unrolled version (200 S-boxes): 1 cycle / block, at least 200 kGE

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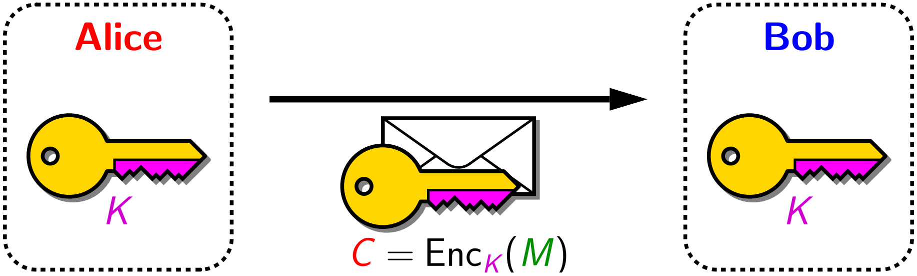

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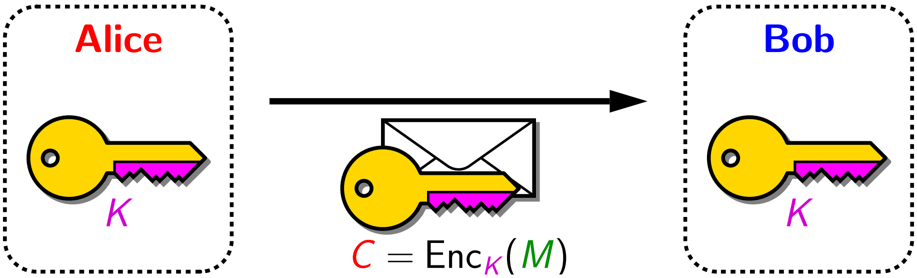

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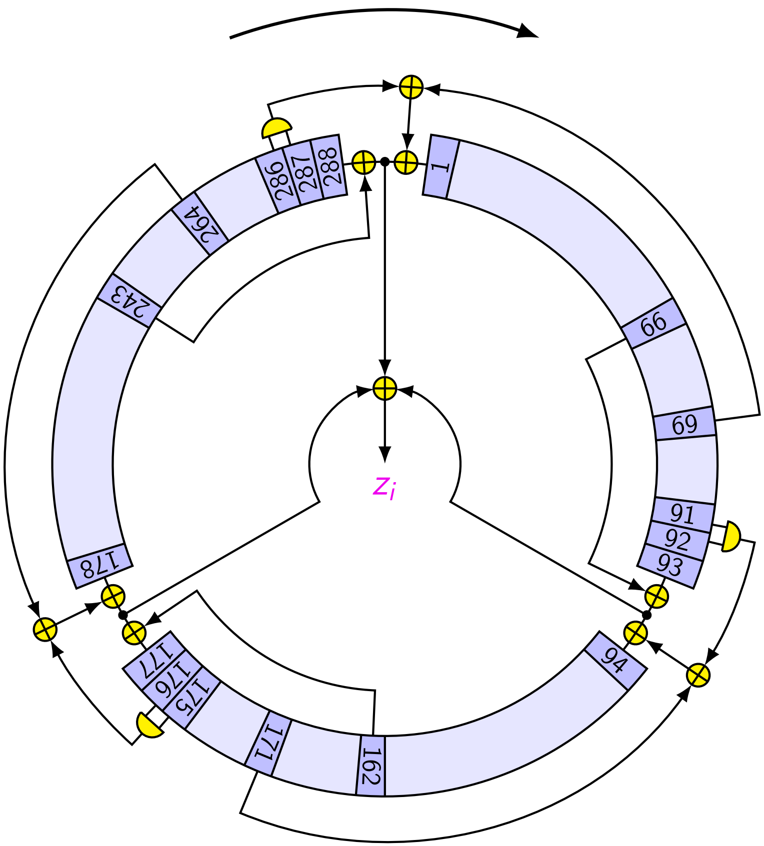
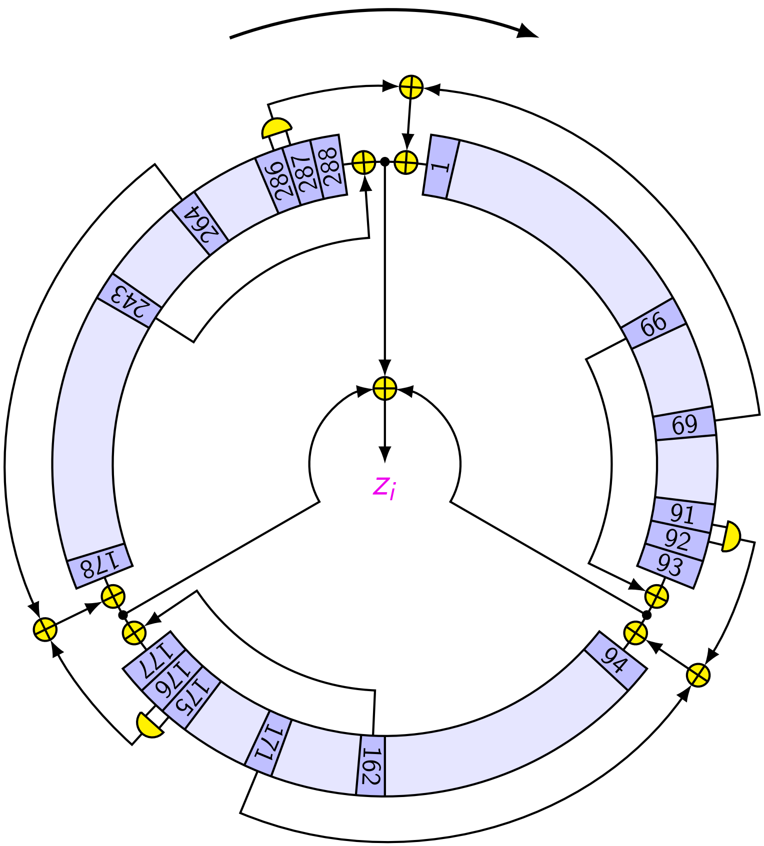
- Stream cipher:
  - generate a pseudorandom keystream Z using a PRNG initialized by the key K and a random initialization vector (IV)
  - use Z to mask the message: $C = M \oplus Z$ and $M = C \oplus Z$ ($\oplus$ is XOR)

Trivium [De Cannière & Preneel, 2005]



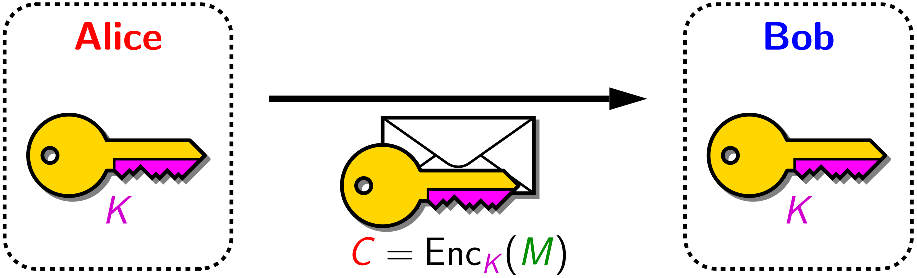
- Part of the eSTREAM portfolio (low-area hardware ciphers)
- Key size: 80 bits
- IV size: 80 bits
- 288-bit circular shift register, plus a few XOR and AND gates

Trivium [De Cannière & Preneel, 2005]



- Part of the eSTREAM portfolio (low-area hardware ciphers)
- Key size: 80 bits
- IV size: 80 bits
- 288-bit circular shift register, plus a few XOR and AND gates
- Serial version:
  - 1 keystream bit / clock cycle
  - 2.6 kGE
- Parallel version:
  - up to 64 bits / clock cycle
  - 4.9 kGE

Public-key encryption



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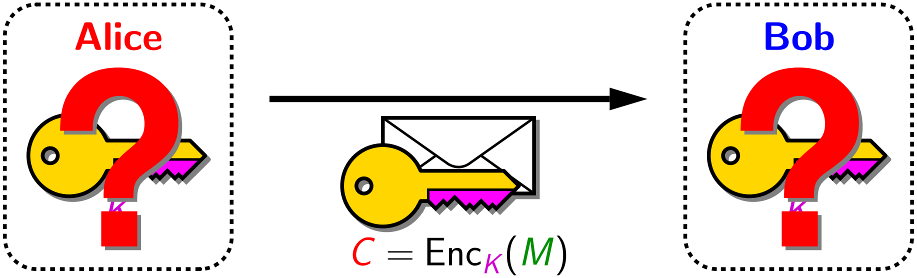

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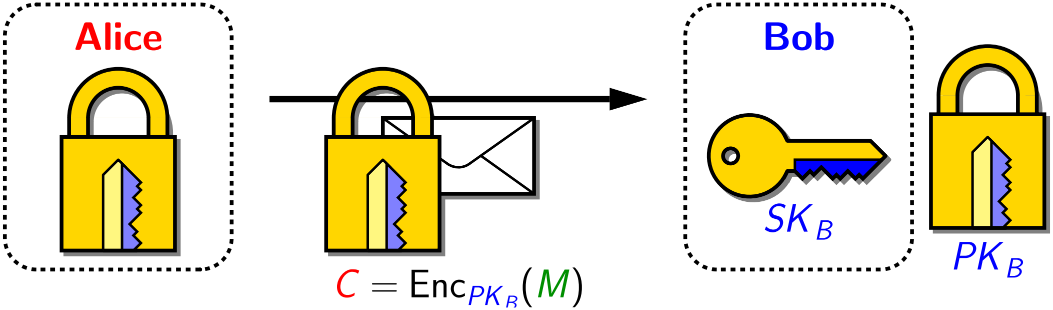

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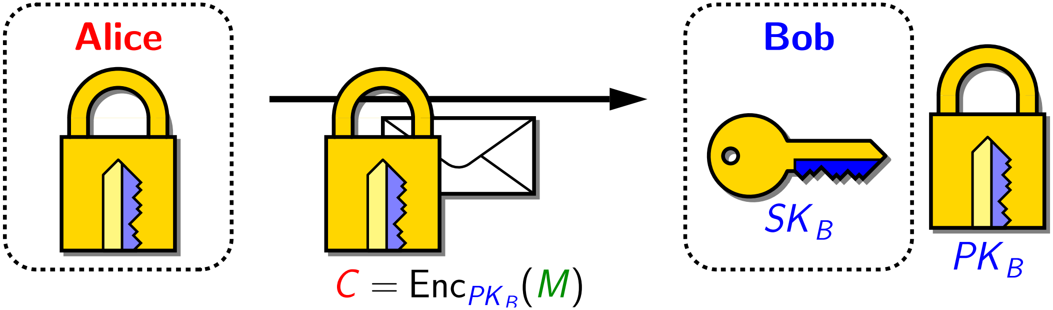

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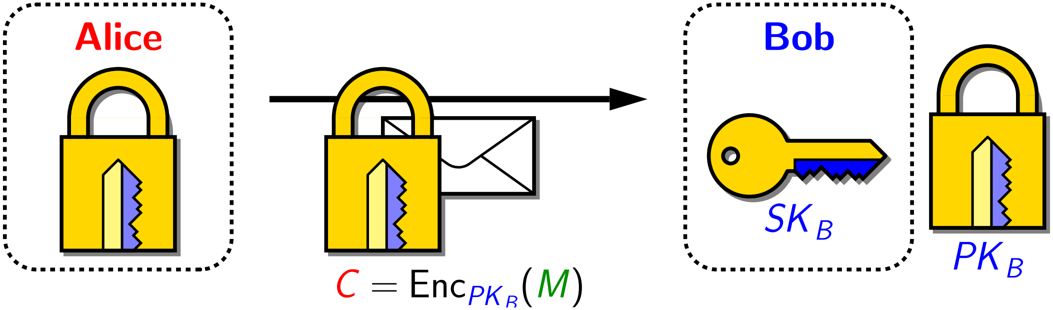

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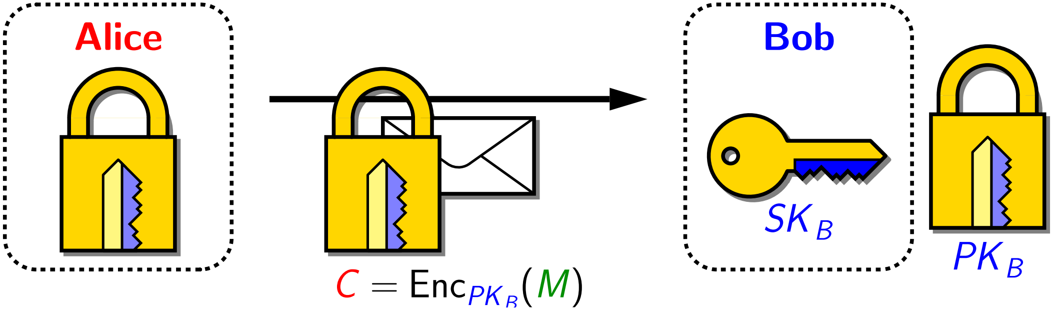

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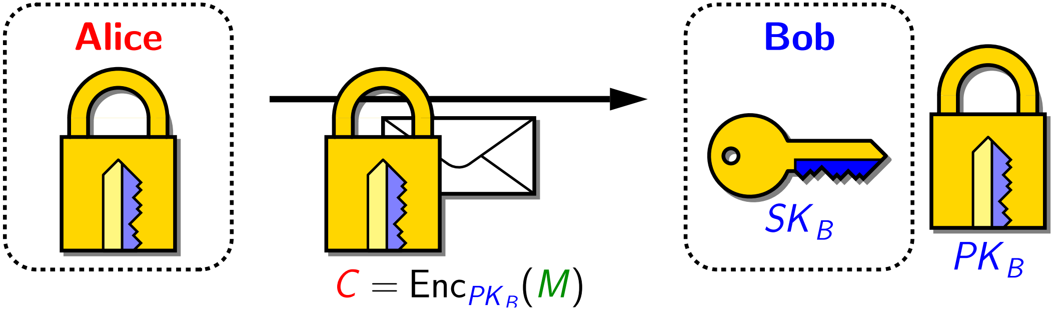

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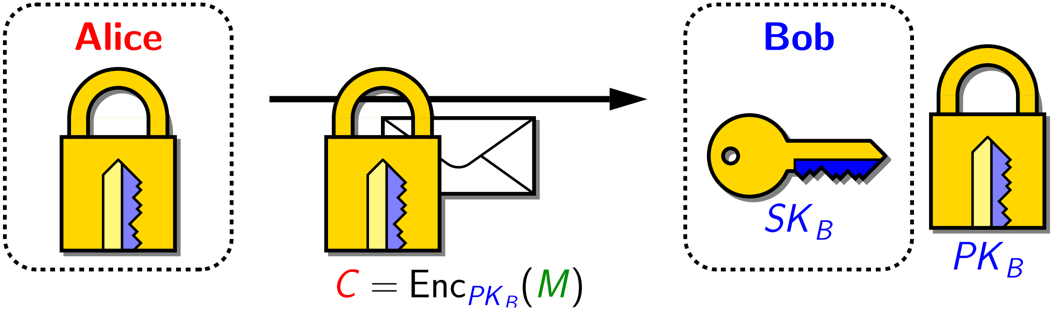

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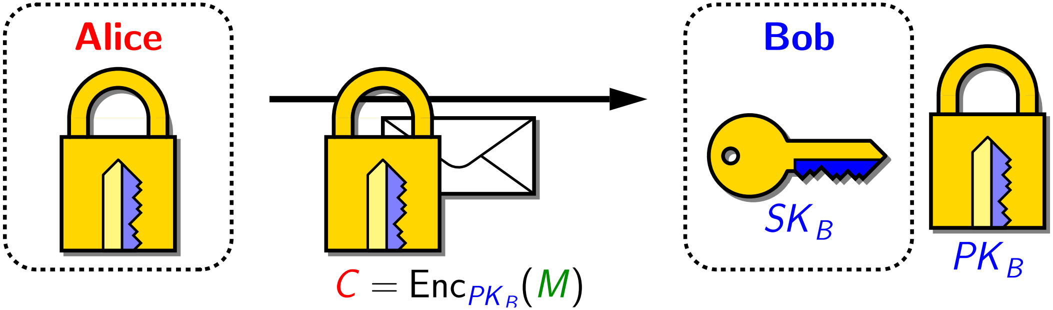

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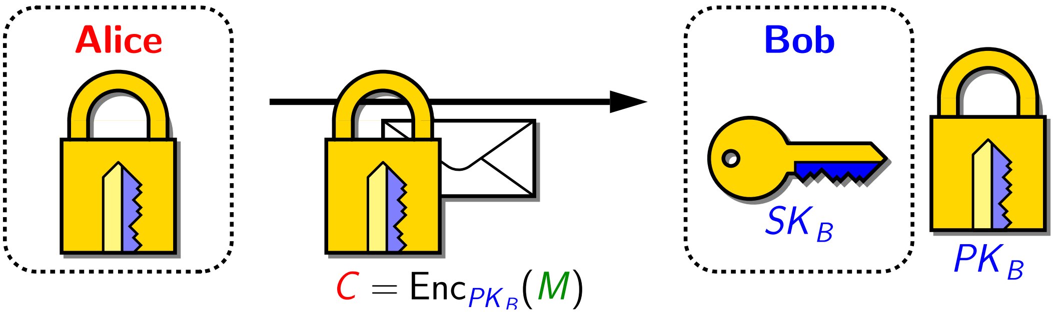

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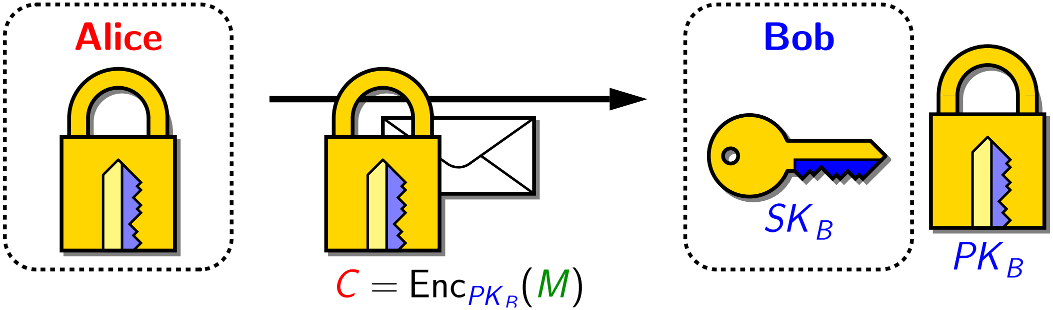

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Outline

- Some encryption mechanisms
- Elliptic curve cryptography
- Scalar multiplication
- Elliptic curve arithmetic
- Finite field arithmetic

A primer on elliptic curves

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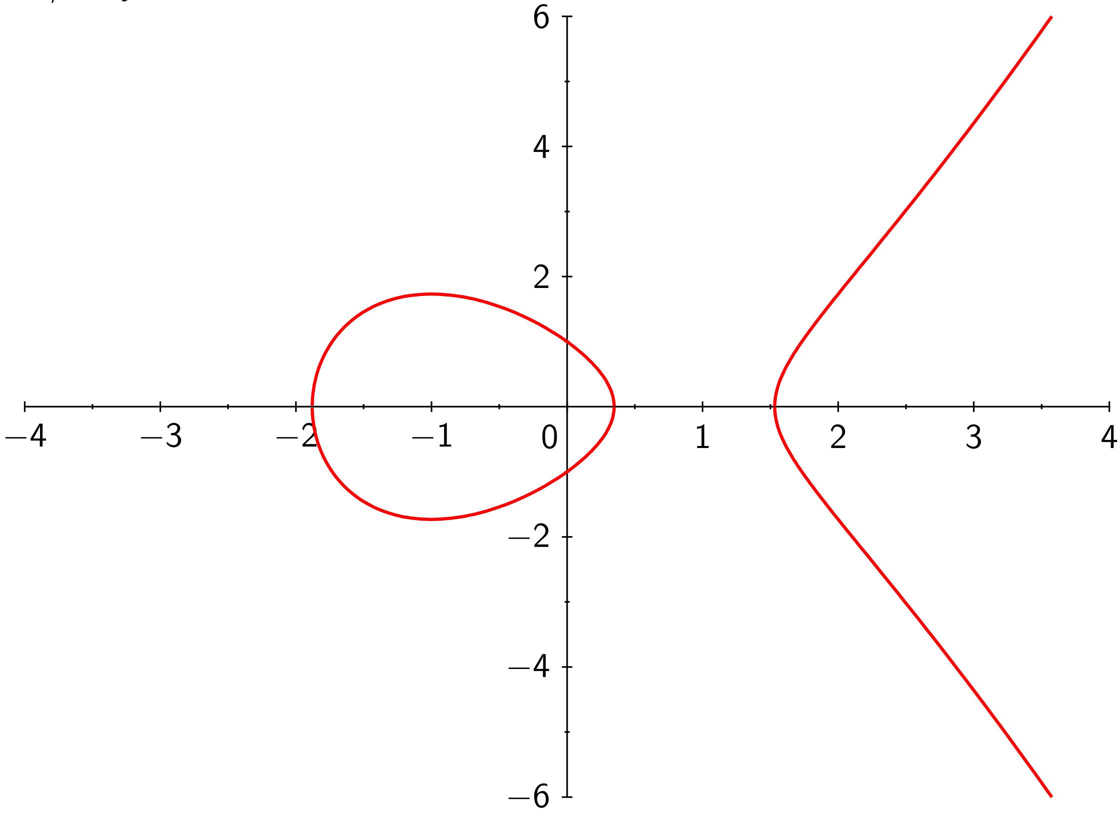
- Additive group law: $E(K)$ is an abelian group
  - addition via the "chord and tangent" method
  - $\mathcal{O}$ is the neutral element

Elliptic curves and group law

$E/\mathbb{R} : y^2 = x^3 - 3x + 1$



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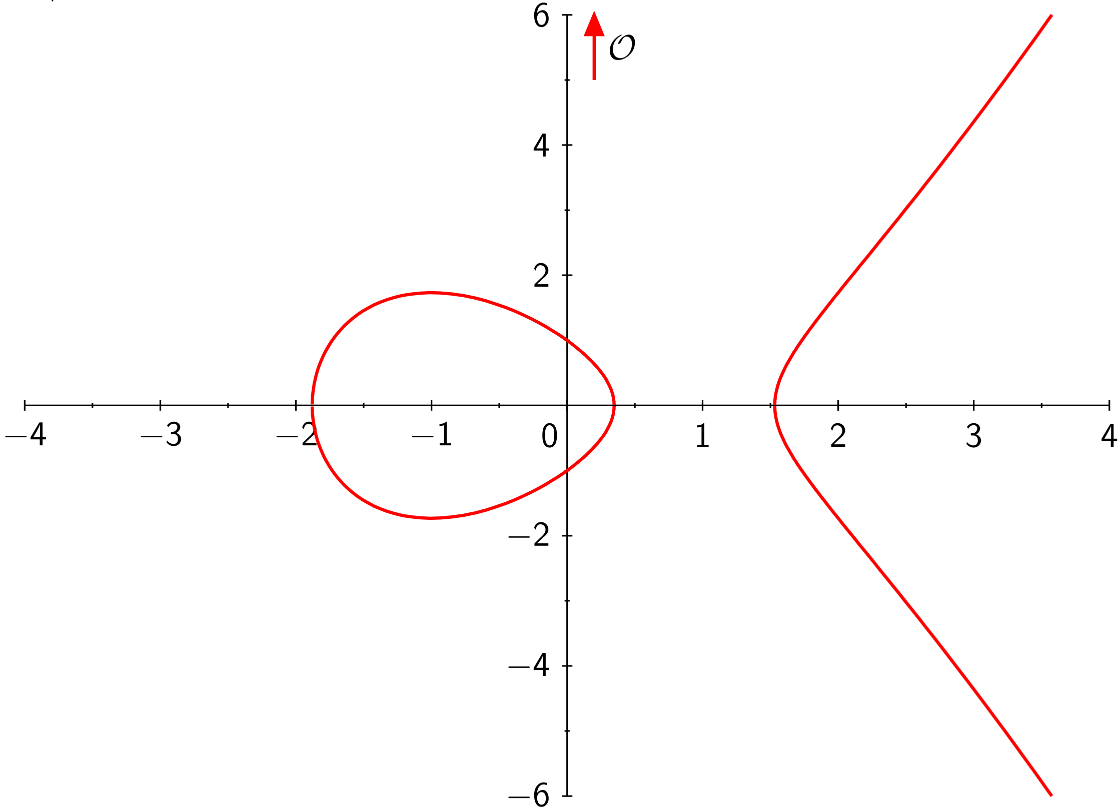

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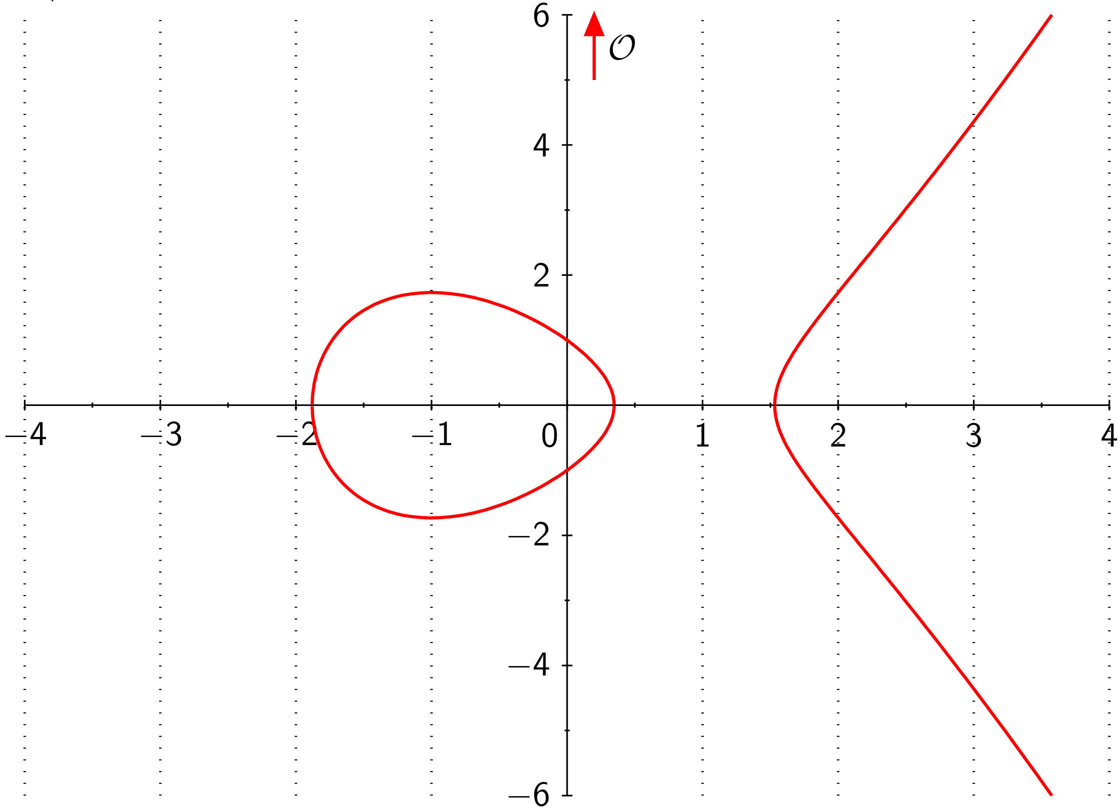

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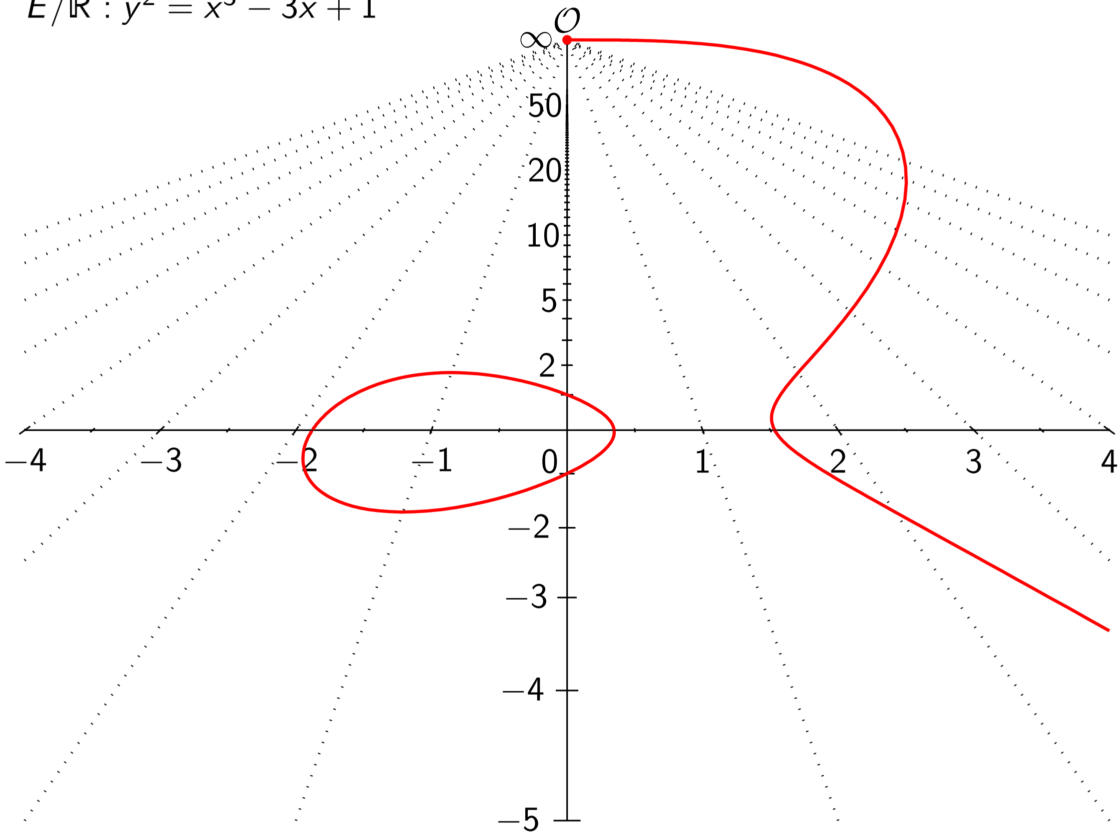

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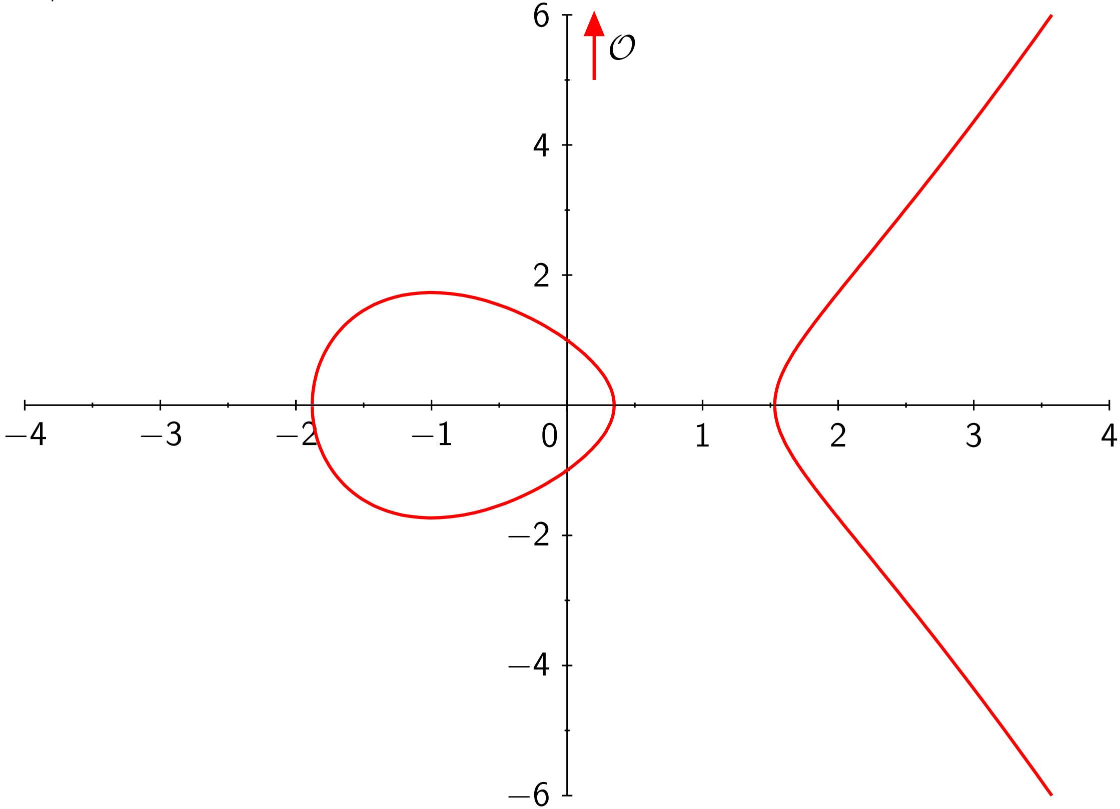

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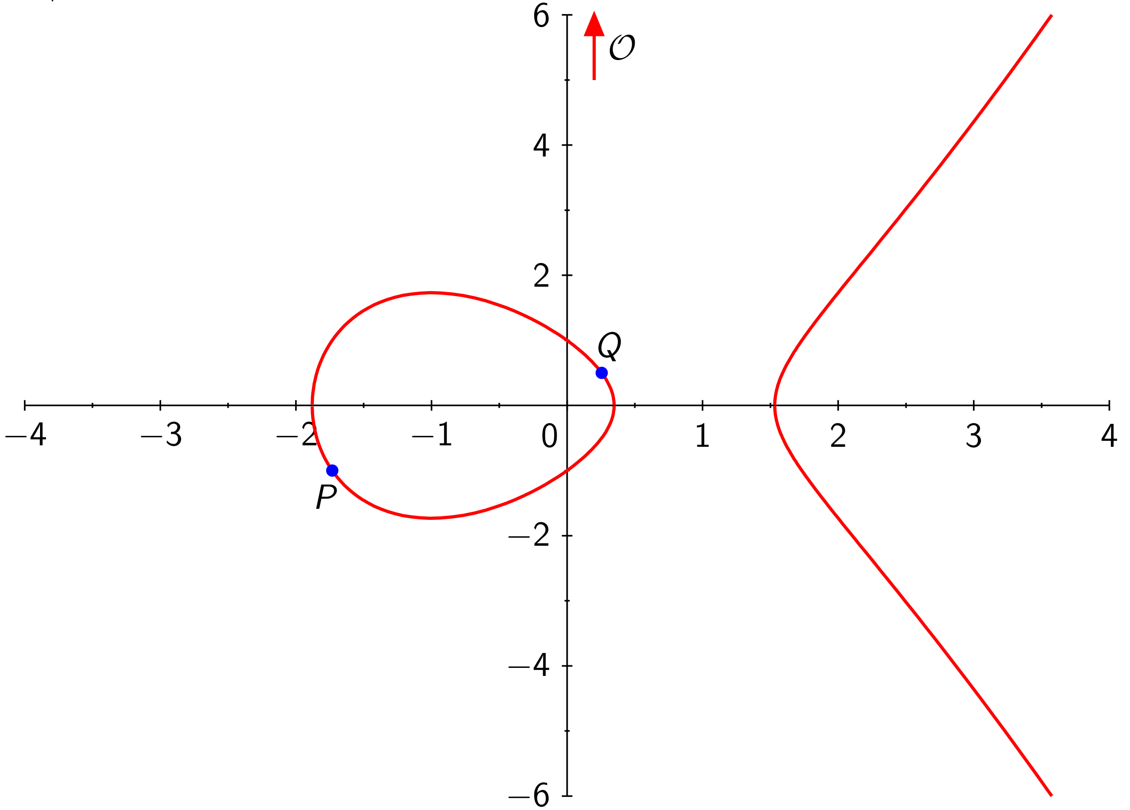

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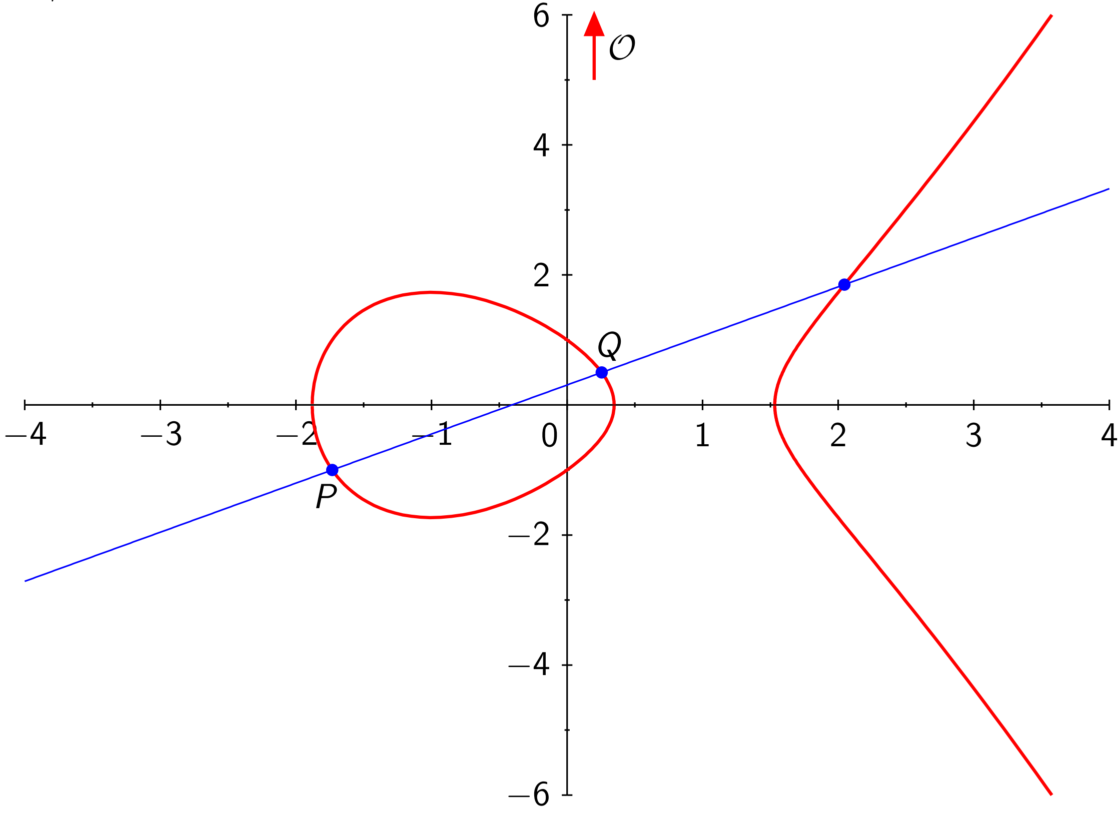

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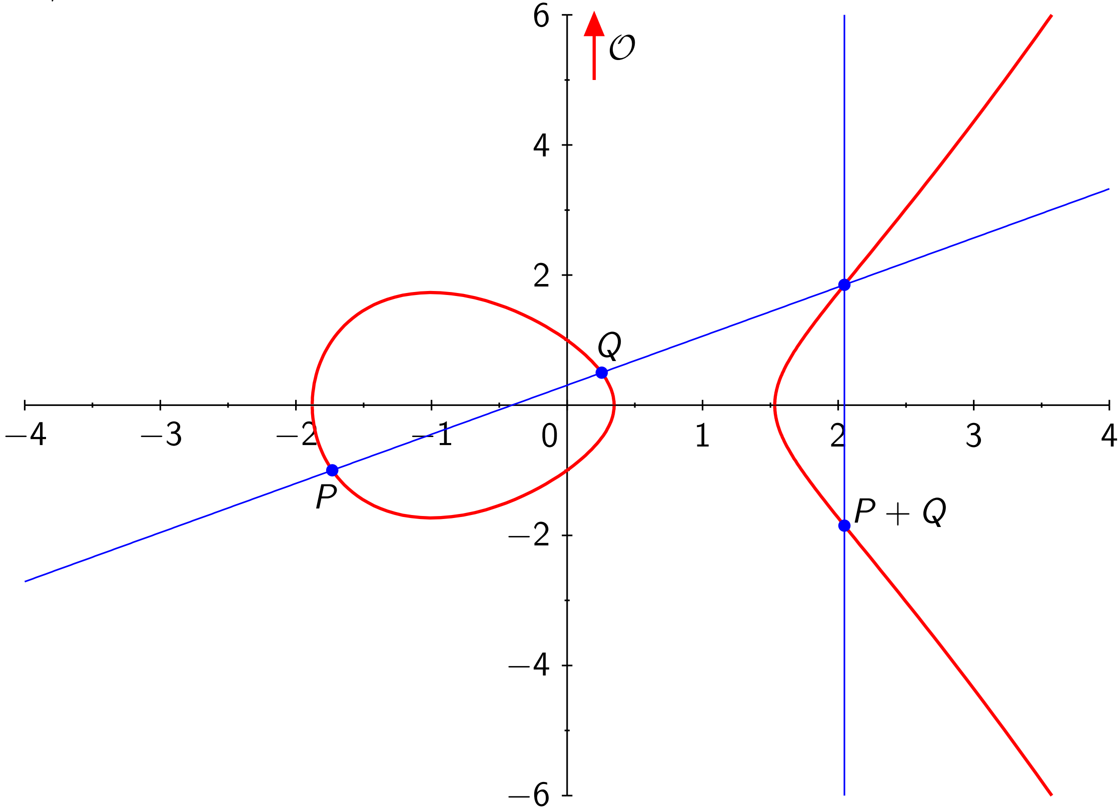

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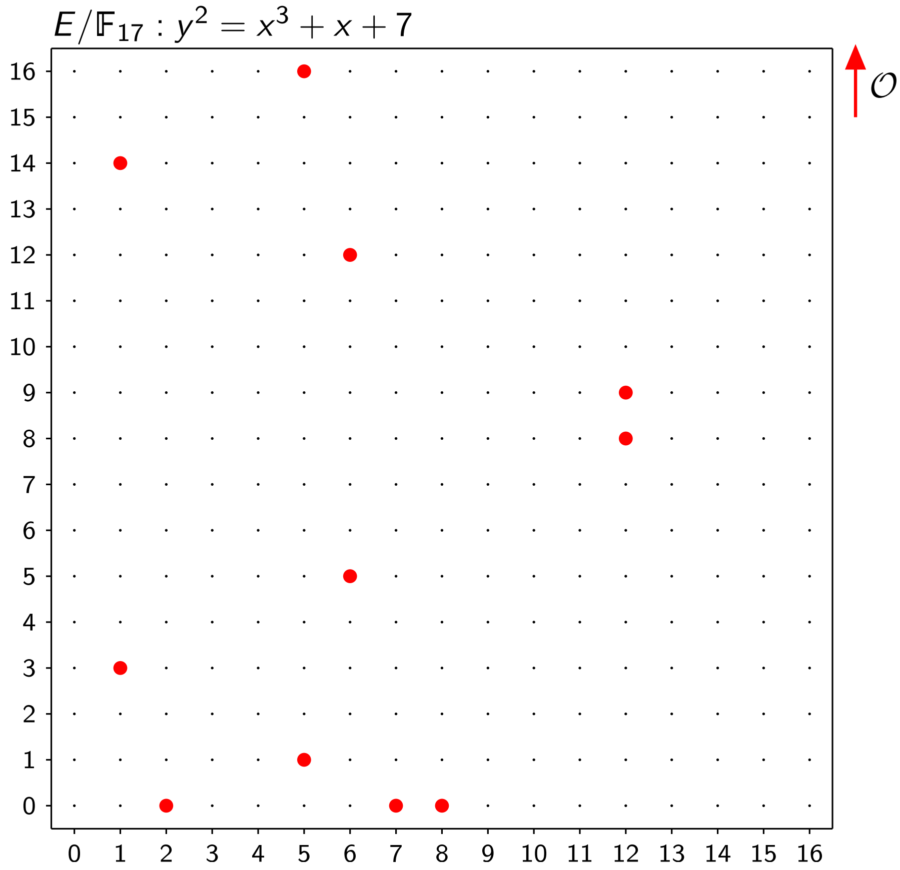

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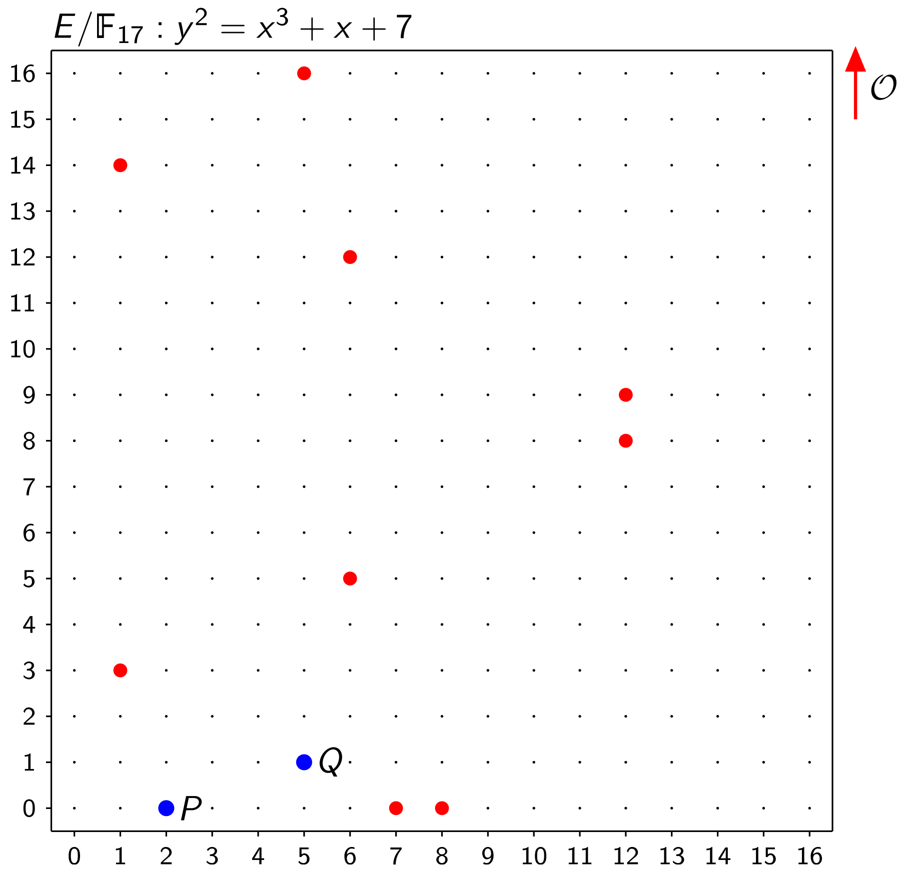

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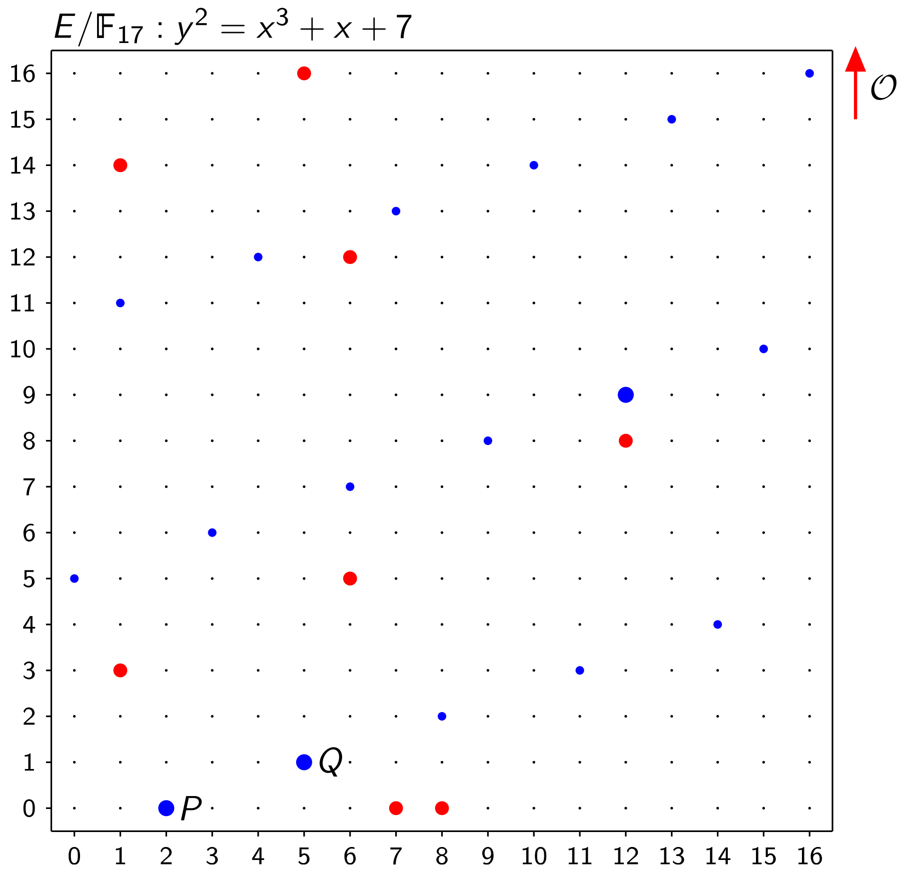

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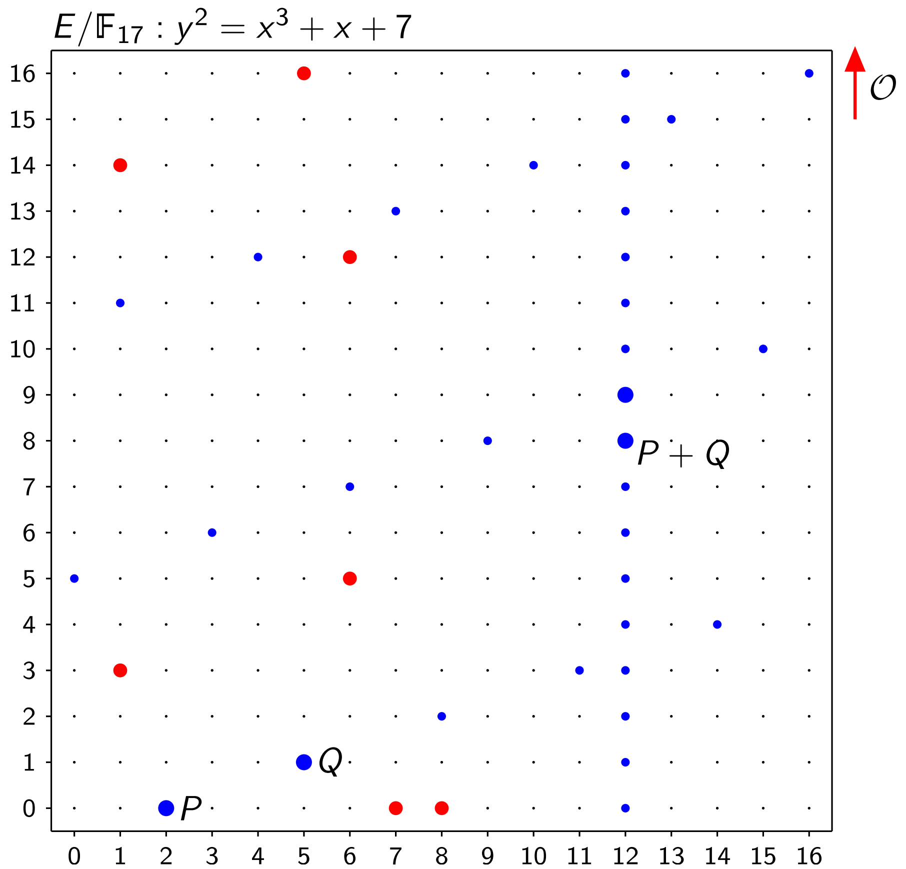

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- The inverse map is the so-called discrete logarithm (in base P):
  $$\begin{array}{rcccll} \mathrm{dlog}_P = \exp_P^{-1} & : & \mathbb{G} & \longrightarrow & \mathbb{Z}/\ell\mathbb{Z} & \\ & & Q & \longmapsto & k & \text{such that } Q = kP \end{array}$$

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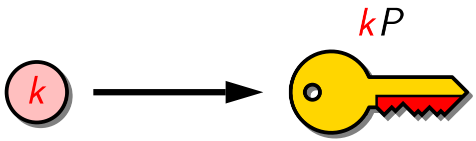

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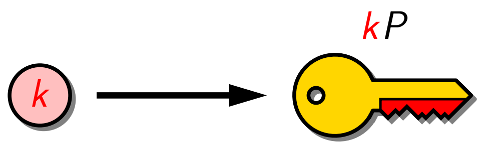

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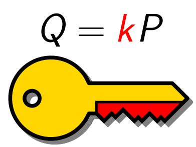

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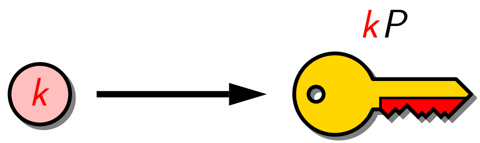

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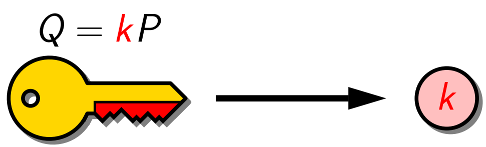

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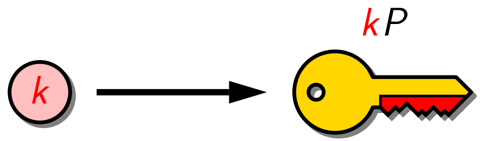

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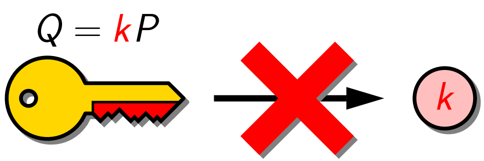

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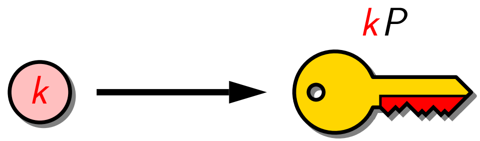

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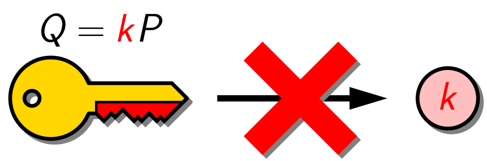

- That's a one-way function $\Rightarrow$ public-key cryptography!

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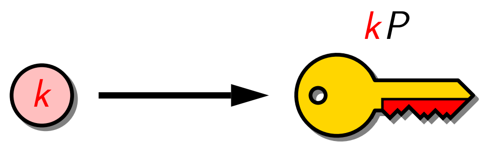

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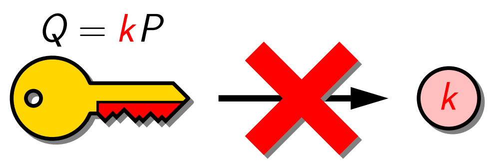

- That's a one-way function $\Rightarrow$ public-key cryptography!
  - secret key: an integer k in $\mathbb{Z}/\ell\mathbb{Z}$
  - public key: the point kP in $\mathbb{G} \subseteq E(\mathbb{F}_q)$

Example protocol: EC Diffie–Hellman key exchange

- Alice and Bob want to establish a secure communication channel

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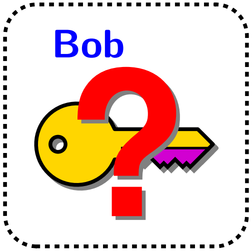

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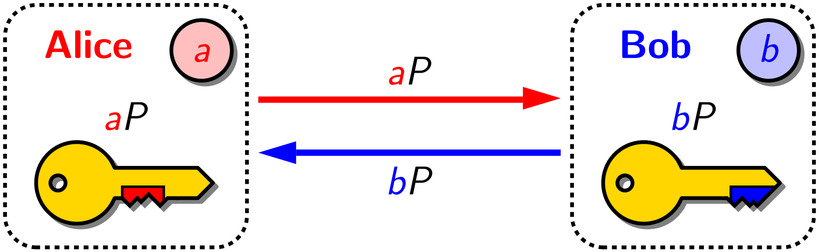

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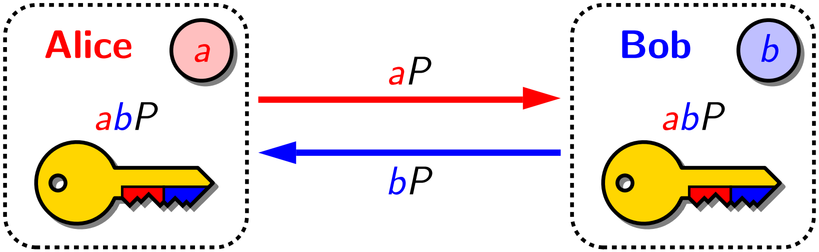

Central operation: the scalar multiplication

- Elliptic curve Diffie–Hellman (ECDH):
  - Alice: $Q_A \leftarrow aP$ and $K \leftarrow aQ_B$ (2 scalar mults)
  - Bob: $Q_B \leftarrow bP$ and $K \leftarrow bQ_A$ (2 scalar mults)

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- Several algorithmic and arithmetic layers:
  - scalar multiplication
  - elliptic curve arithmetic (point addition, point doubling, etc.)
  - finite field arithmetic (addition, multiplication, inversion, etc.)

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  - PAVOIS project: ECC cryptoprocessor designed to evaluate algorithmic and arithmetic protections against side-channel attacks [See A. Tisserand's talk]

Outline

- Some encryption mechanisms
- Elliptic curve cryptography
- Scalar multiplication
- Elliptic curve arithmetic
- Finite field arithmetic

Scalar multiplication

- Given k in $\mathbb{Z}/\ell\mathbb{Z}$ and P in $\mathbb{G} \subseteq E(\mathbb{F}_q)$, we want to compute

$$kP = \underbrace{P + P + \ldots + P}_{k \text{ times}}$$

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- Size of ℓ (and k) for crypto applications: from 250 to 500 bits
- Repeated addition, in $O(k)$ complexity, is out of the question!

Double-and-add algorithm

- Available operations on $E(\mathbb{F}_q)$:
  - point addition: $(Q, R) \mapsto Q + R$
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  - add P if the corresponding bit of k is 1
  - same principle as binary exponentiation

Double-and-add algorithm

- Denoting by $(k_{n-1} \ldots k_1 k_0)_2$, with $n = \lceil \log_2 \ell \rceil$, the binary expansion of k:

function scalar-mult(k, P)**:**
$T \leftarrow \mathcal{O}$
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$$T = \qquad = \quad \mathcal{O}$$

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- Example: $k = 431 = (\underline{1}10101111)_2$

$T = \quad P \quad\quad = \quad P$

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- Example: $k = 431 = (1\underline{1}0101111)_2$

$$T = \quad P \cdot 2 \quad = \quad 2P$$

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      $T \leftarrow T + P$
  return T

- Example: $k = 431 = (1\underline{1}0101111)_2$

$$T = \quad P \cdot 2 + P \quad = \quad 3P$$

Double-and-add algorithm

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        if $k_i = 1$**:**
            $T \leftarrow T + P$
    return T

- Example: $k = 431 = (11\underline{0}101111)_2$

$$T = \quad (P \cdot 2 + P) \cdot 2 \quad = \quad 6P$$

Double-and-add algorithm

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  return T

- Example: $k = 431 = (110\underline{1}01111)_2$

$$T = \quad (P \cdot 2 + P) \cdot 2^2 \qquad = \quad 12P$$

Double-and-add algorithm

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- Example: $k = 431 = (110\underline{1}01111)_2$

$$T = ((P \cdot 2 + P) \cdot 2^2 + P) \cdot 2 = 26P$$

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- Example: $k = 431 = (110101111)_2$

$$T = ((P \cdot 2 + P) \cdot 2^2 + P) \cdot 2^2 = 52P$$

Double-and-add algorithm

- Denoting by $(k_{n-1} \dots k_1 k_0)_2$, with $n = \lceil \log_2 \ell \rceil$, the binary expansion of k:

$$
\begin{array}{l}
\textbf{function } \text{scalar-mult}(k, P)\textbf{:} \\
\quad T \leftarrow \mathcal{O} \\
\quad \textbf{for } i \leftarrow n-1 \textbf{ downto } 0\textbf{:} \\
\quad\quad T \leftarrow 2T \\
\quad\quad \textbf{if } k_i = 1\textbf{:} \\
\quad\quad\quad T \leftarrow T + P \\
\quad \textbf{return } T
\end{array}
$$

- Example: $k = 431 = (110101\underline{1}11)_2$

$$T = \quad ((P \cdot 2 + P) \cdot 2^2 + P) \cdot 2^2 + P \qquad = \quad 53P$$

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- Example: $k = 431 = (110101\underline{1}11)_2$

$$T = (((P \cdot 2 + P) \cdot 2^2 + P) \cdot 2^2 + P) \cdot 2 \qquad = 106P$$

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- Example: $k = 431 = (110101\underline{1}1)_2$

$$T = ((((P \cdot 2 + P) \cdot 2^2 + P) \cdot 2^2 + P) \cdot 2 + P) \cdot 2 \qquad = 214P$$

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$$T = (((((P \cdot 2 + P) \cdot 2^2 + P) \cdot 2^2 + P) \cdot 2 + P) \cdot 2 + P) \cdot 2 \qquad = 430P$$

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```
function scalar-mult(k, P):
    T ← O
    for i ← n − 1 downto 0:
        T ← 2T
        if k_i = 1:
            T ← T + P
    return T
```

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- Complexity in $O(n) = O(\log_2 \ell)$ operations over $E(\mathbb{F}_q)$:
  - $n-1$ doublings, and
  - $n/2$ additions on average

Windowed method

- Consider 2^w-ary expansion of k: i.e., split k into w-bit chunks

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  - $2^{w-1} - 1$ additions

Windowed method

- Consider 2^w-ary expansion of k: i.e., split k into w-bit chunks
- Precompute $2P$, $3P$, $\ldots$, $(2^w - 1)P$:
  - $2^{w-1} - 1$ doublings, and
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- Example with $w = 3$: $k = 431$

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- Example with $w = 3$: $k = 431 = (110\,101\,111)_2$

Windowed method

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- Example with $w = 3$: $k = 431 = (110\,101\,111)_2 = (657)_{2^3}$

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- Example with $w = 3$: $k = 431 = (110\,101\,111)_2 = (657)_{2^3}$

$$T = \qquad = \quad \mathcal{O}$$

Windowed method

- Consider 2^w-ary expansion of k: i.e., split k into w-bit chunks
- Precompute $2P, 3P, \ldots, (2^w - 1)P$:
  - $2^{w-1} - 1$ doublings, and
  - $2^{w-1} - 1$ additions
- Example with $w = 3$: $k = 431 = (\underline{110}\,101\,111)_2 = (\underline{6}57)_{2^3}$

$$T = 6P \qquad = \quad 6P$$

Windowed method

- Consider 2^w-ary expansion of k: i.e., split k into w-bit chunks
- Precompute $2P, 3P, \ldots, (2^w - 1)P$:
  - $2^{w-1} - 1$ doublings, and
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- Example with $w = 3$: $k = 431 = (110\,\underline{101}\,111)_2 = (6\underline{5}7)_{2^3}$

$$T = 6P \cdot 2^3 = 48P$$

Windowed method

- Consider 2^w-ary expansion of k: i.e., split k into w-bit chunks
- Precompute $2P, 3P, \ldots, (2^w - 1)P$:
  - $2^{w-1} - 1$ doublings, and
  - $2^{w-1} - 1$ additions
- Example with $w = 3$: $k = 431 = (110\,\underline{101}\,111)_2 = (6\underline{5}7)_{2^3}$

$$T = 6P \cdot 2^3 + 5P \qquad = 53P$$

Windowed method

- Consider 2^w-ary expansion of k: i.e., split k into w-bit chunks
- Precompute $2P, 3P, \ldots, (2^w - 1)P$:
  - $2^{w-1} - 1$ doublings, and
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- Example with $w = 3$: $k = 431 = (110\,101\,\underline{111})_2 = (65\underline{7})_{2^3}$

$$T = (6P \cdot 2^3 + 5P) \cdot 2^3 \qquad = 424P$$

Windowed method

- Consider 2^w-ary expansion of k: i.e., split k into w-bit chunks
- Precompute $2P, 3P, \ldots, (2^w - 1)P$:
  - $2^{w-1} - 1$ doublings, and
  - $2^{w-1} - 1$ additions
- Example with $w = 3$: $k = 431 = (110\,101\,\underline{111})_2 = (65\underline{7})_{2^3}$

$$T = (6P \cdot 2^3 + 5P) \cdot 2^3 + 7P = 431P$$

Windowed method

- Consider 2^w-ary expansion of k: i.e., split k into w-bit chunks
- Precompute $2P, 3P, \ldots, (2^w - 1)P$:
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- Example with $w = 3$: $k = 431 = (110\,101\,111)_2 = (657)_{2^3}$

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Windowed method

- Consider 2^w-ary expansion of k: i.e., split k into w-bit chunks
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- Example with $w = 3$: $k = 431 = (110\,101\,111)_2 = (657)_{2^3}$

$$T = (6P \cdot 2^3 + 5P) \cdot 2^3 + 7P = 431P$$

- Complexity:
  - $n - w$ doublings, and
  - $(1 - 2^{-w})n/w$ additions on average

Windowed method

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- Example with $w = 3$: $k = 431 = (110\,101\,111)_2 = (657)_{2^3}$

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Windowed method

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- Example with $w = 3$: $k = 431 = (110\,101\,111)_2 = (657)_{2^3}$

$$T = (6P \cdot 2^3 + 5P) \cdot 2^3 + 7P = 431P$$

- Complexity:
  - $n - w$ doublings, and
  - $(1 - 2^{-w})n/w$ additions on average
- Select w carefully so that precomputation cost does not become predominant
- Sliding window variant: half as many precomputations

Security issues

- Back to the double-and-add algorithm:

function scalar-mult(k, P)**:**
  $T \leftarrow \mathcal{O}$
  for $i \leftarrow n-1$ **downto** 0**:**
    $T \leftarrow 2T$
    if $k_i = 1$**:**
      $T \leftarrow T + P$
  return T

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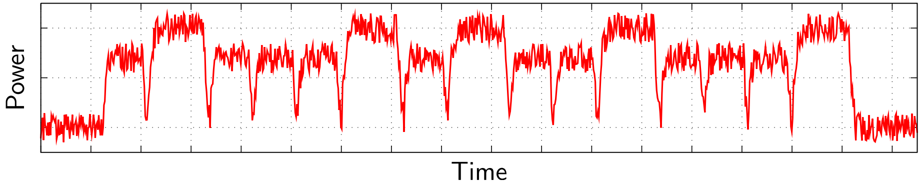
- At step i, point addition $T \leftarrow T + P$ is computed if and only if $k_i = 1$
  - careful timing analysis will reveal Hamming weight of secret k
  - simple power analysis (SPA) will leak bits of k



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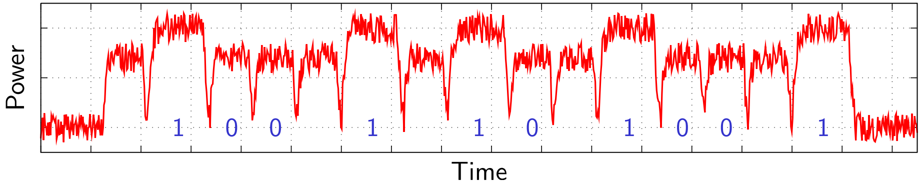

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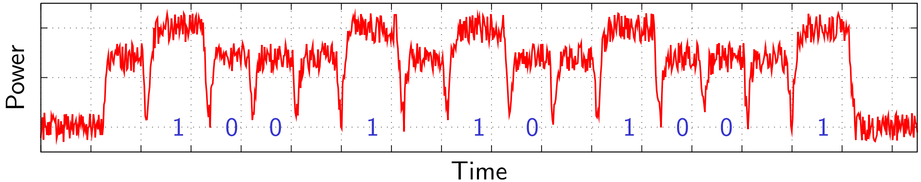

- Use double-and-add-always algorithm?

Security issues

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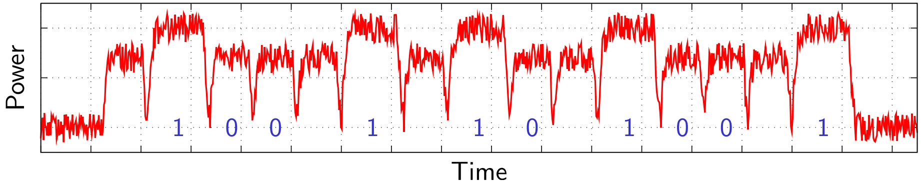

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  - the result of the point addition is used if and only if $k_i = 1$

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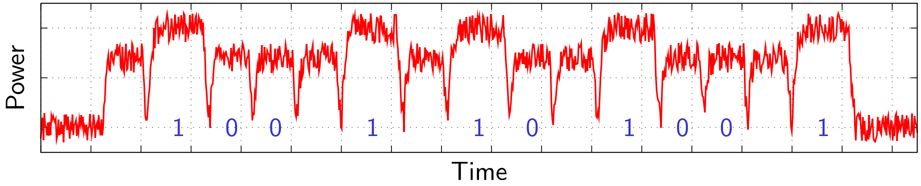

- Use double-and-add-always algorithm?
  - the result of the point addition is used if and only if $k_i = 1$

  $\Rightarrow$ vulnerable to fault attacks [See A. Tisserand's lecture]

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

```
function scalar-mult(k, P):
    T_0 ← O
    T_1 ← P
    for i ← n − 1 downto 0:
        if k_i = 1:
            T_0 ← T_0 + T_1
            T_1 ← 2T_1
        else:
            T_1 ← T_0 + T_1
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- Properties:

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- Properties:
  - perform one addition and one doubling at each step

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- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step

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    return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$

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```

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

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function scalar-mult(k, P):
    T_0 ← O
    T_1 ← P
    for i ← n − 1 downto 0:
        if k_i = 1:
            T_0 ← T_0 + T_1
            T_1 ← 2 T_1
        else:
            T_1 ← T_0 + T_1
            T_0 ← 2 T_0
    return T_0
```

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (10011)_2$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

```
function scalar-mult(k, P):
    T_0 ← O
    T_1 ← P
    for i ← n − 1 downto 0:
        if k_i = 1:
            T_0 ← T_0 + T_1
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            T_1 ← T_0 + T_1
            T_0 ← 2T_0
    return T_0
```

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (10011)_2$

$$T_0 = \quad = \mathcal{O}$$
$$T_1 = P \quad = P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

  function scalar-mult(k, P)**:**
  $T_0 \leftarrow \mathcal{O}$
  $T_1 \leftarrow P$
  for $i \leftarrow n-1$ **downto** 0**:**
  if $k_i = 1$**:**
  $T_0 \leftarrow T_0 + T_1$
  $T_1 \leftarrow 2T_1$
  else:
  $T_1 \leftarrow T_0 + T_1$
  $T_0 \leftarrow 2T_0$
  return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$

- Example: $k = 19 = (\underline{1}0011)_2$

$$T_0 = \quad = \mathcal{O}$$
$$T_1 = P \quad = P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

function scalar-mult(k, P)**:**
  $T_0 \leftarrow \mathcal{O}$
  $T_1 \leftarrow P$
  for $i \leftarrow n-1$ **downto** 0**:**
    if $k_i = 1$**:**
      $T_0 \leftarrow T_0 + T_1$
      $T_1 \leftarrow 2T_1$
    else:
      $T_1 \leftarrow T_0 + T_1$
      $T_0 \leftarrow 2T_0$
  return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (\underline{1}0011)_2$

$$T_0 = P \qquad = P$$
$$T_1 = P \qquad = P$$

The Montgomery ladder

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function scalar-mult(k, P)**:**
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    if $k_i = 1$**:**
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      $T_1 \leftarrow 2T_1$
    else:
      $T_1 \leftarrow T_0 + T_1$
      $T_0 \leftarrow 2T_0$
  return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (\underline{1}0011)_2$

$$T_0 = P \qquad = P$$
$$T_1 = P \cdot 2 \qquad = 2P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

function scalar-mult(k, P)**:**

$T_0 \leftarrow \mathcal{O}$

$T_1 \leftarrow P$

for $i \leftarrow n-1$ **downto** 0**:**

if $k_i = 1$**:**

$T_0 \leftarrow T_0 + T_1$

$T_1 \leftarrow 2T_1$

else:

$T_1 \leftarrow T_0 + T_1$

$T_0 \leftarrow 2T_0$

return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$

- Example: $k = 19 = (1\underline{0}011)_2$

$$T_0 = P \qquad = P$$

$$T_1 = P \cdot 2 \qquad = 2P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

function scalar-mult(k, P)**:**
  $T_0 \leftarrow \mathcal{O}$
  $T_1 \leftarrow P$
  for $i \leftarrow n-1$ **downto** 0**:**
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      $T_0 \leftarrow T_0 + T_1$
      $T_1 \leftarrow 2T_1$
    else:
      $T_1 \leftarrow T_0 + T_1$
      $T_0 \leftarrow 2T_0$
  return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (1\underline{0}011)_2$

$$T_0 = P \qquad = P$$
$$T_1 = P \cdot 2 + P \qquad = 3P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

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function scalar-mult(k, P):
    T_0 ← O
    T_1 ← P
    for i ← n − 1 downto 0:
        if k_i = 1:
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        else:
            T_1 ← T_0 + T_1
            T_0 ← 2T_0
    return T_0
```

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (1\underline{0}011)_2$

$$T_0 = P \cdot 2 = 2P$$
$$T_1 = P \cdot 2 + P = 3P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

```
function scalar-mult(k, P):
    T_0 ← O
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            T_0 ← 2T_0
    return T_0
```

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (10\underline{0}11)_2$

$$T_0 = P \cdot 2 = 2P$$
$$T_1 = P \cdot 2 + P = 3P$$

The Montgomery ladder

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  $T_0 \leftarrow 2T_0$
  return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (10\underline{0}11)_2$

$$T_0 = P \cdot 2 = 2P$$
$$T_1 = P \cdot 2 + P + 2P = 5P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

```
function scalar-mult(k, P):
    T_0 ← O
    T_1 ← P
    for i ← n − 1 downto 0:
        if k_i = 1:
            T_0 ← T_0 + T_1
            T_1 ← 2T_1
        else:
            T_1 ← T_0 + T_1
            T_0 ← 2T_0
    return T_0
```

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (10\underline{0}11)_2$

$$T_0 = P \cdot 2^2 = 4P$$
$$T_1 = P \cdot 2 + P + 2P = 5P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

function scalar-mult(k, P)**:**
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    return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (100\underline{1}1)_2$

$$T_0 = P \cdot 2^2 = 4P$$
$$T_1 = P \cdot 2 + P + 2P = 5P$$

The Montgomery ladder

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function scalar-mult(k, P)**:**
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- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (100\underline{1}1)_2$

$$T_0 = P \cdot 2^2 + 5P = 9P$$
$$T_1 = P \cdot 2 + P + 2P = 5P$$

The Montgomery ladder

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- Properties:
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- Example: $k = 19 = (100\underline{1}1)_2$

$$T_0 = P \cdot 2^2 + 5P = 9P$$
$$T_1 = (P \cdot 2 + P + 2P) \cdot 2 = 10P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

function scalar-mult(k, P)**:**
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    $T_1 \leftarrow P$
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$$\begin{aligned} T_0 &= P \cdot 2^2 + 5P &&= 9P \\ T_1 &= (P \cdot 2 + P + 2P) \cdot 2 &&= 10P \end{aligned}$$

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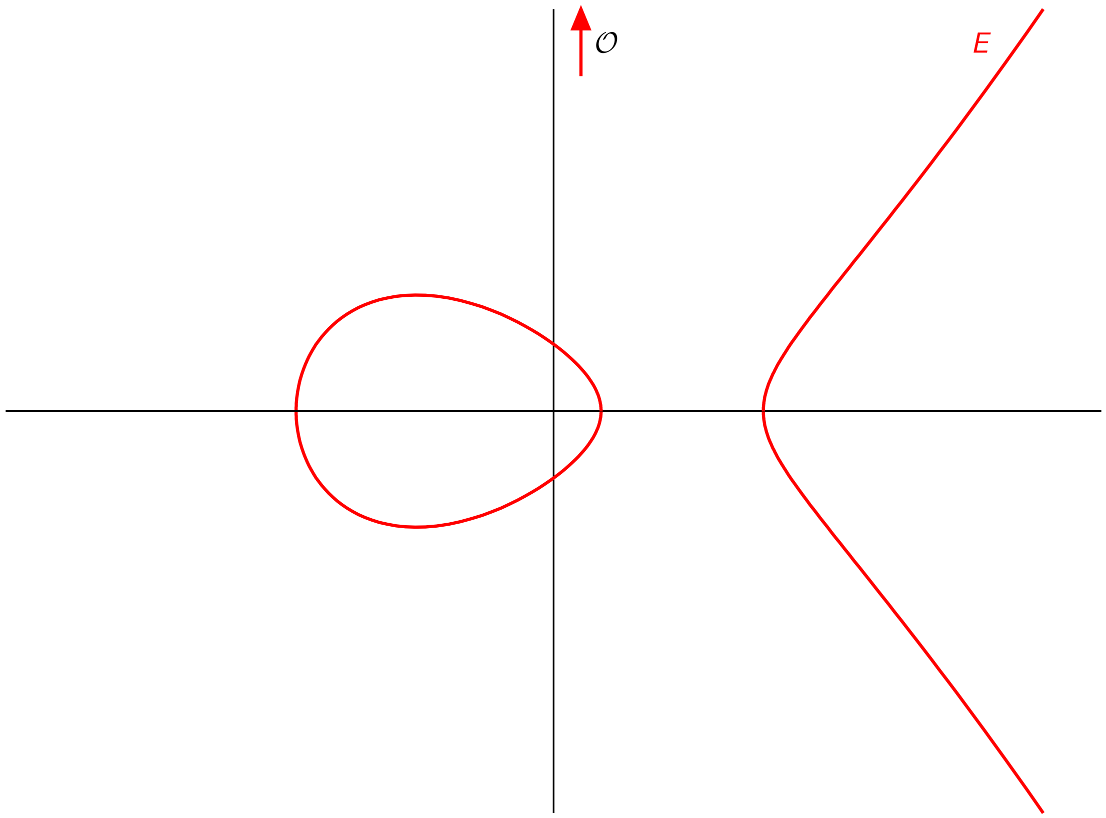
Outline

- Some encryption mechanisms
- Elliptic curve cryptography
- Scalar multiplication
- Elliptic curve arithmetic
- Finite field arithmetic

Addition and doubling



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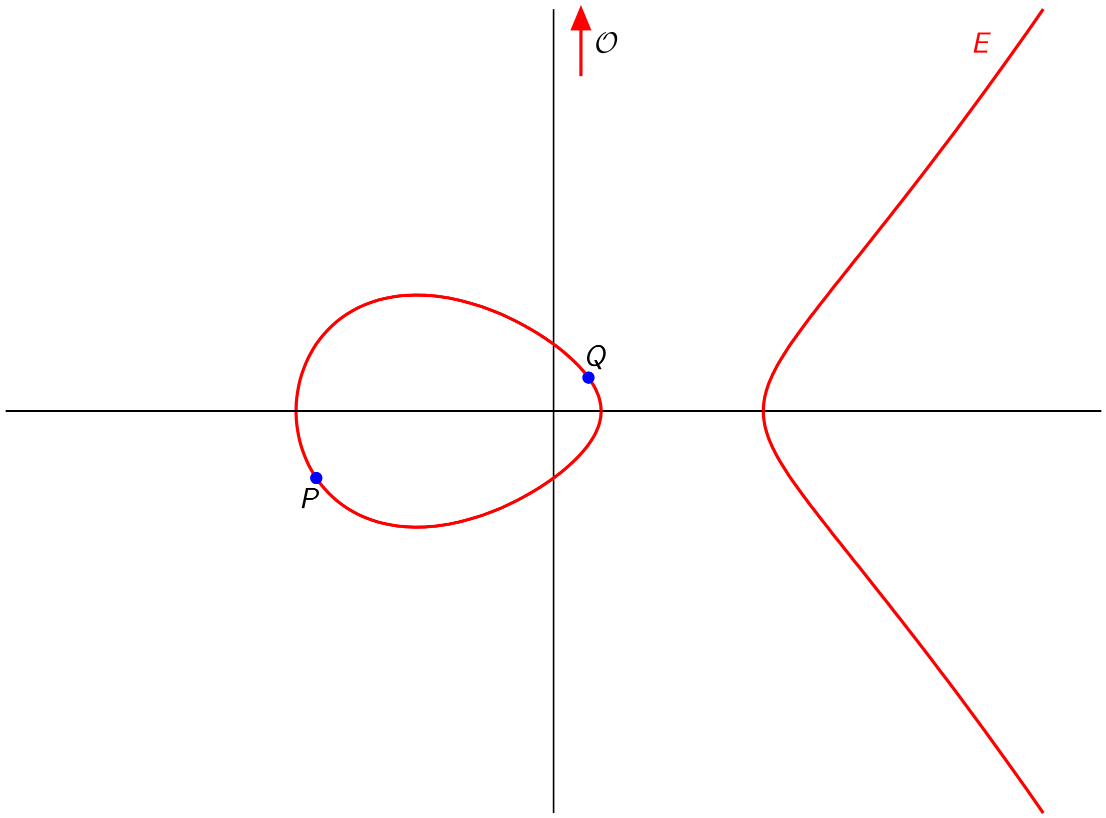

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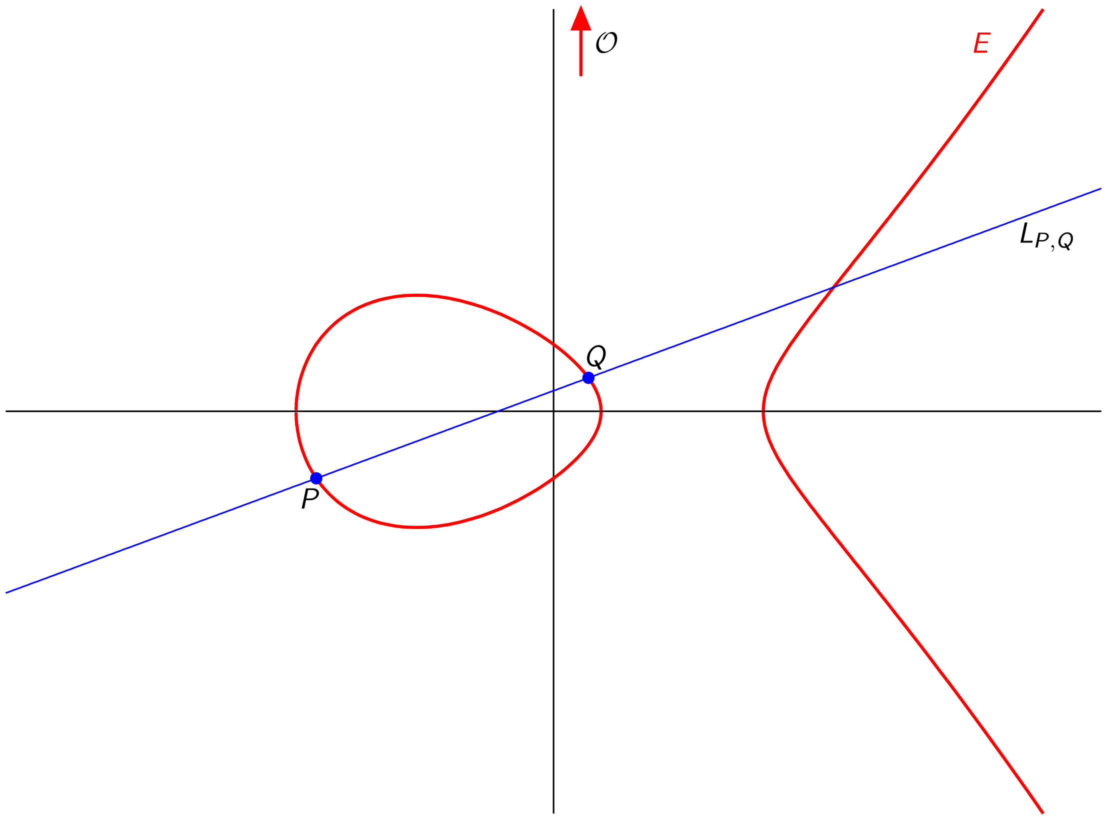

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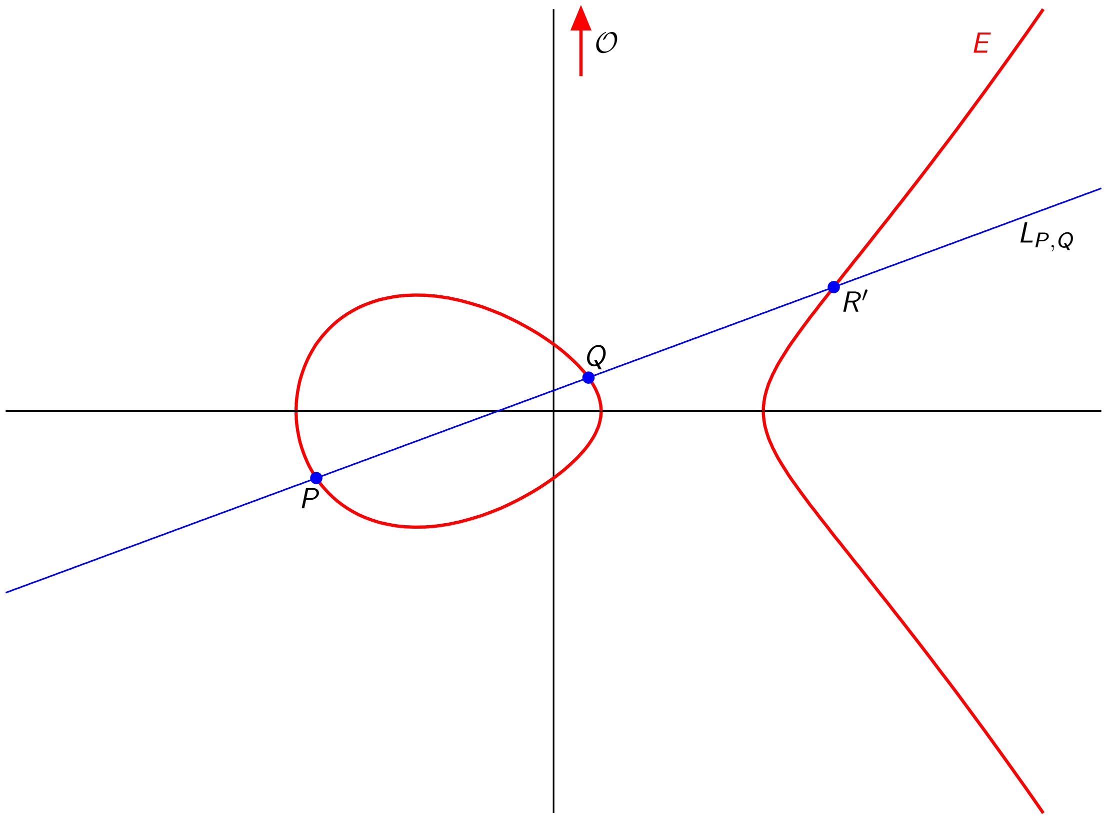

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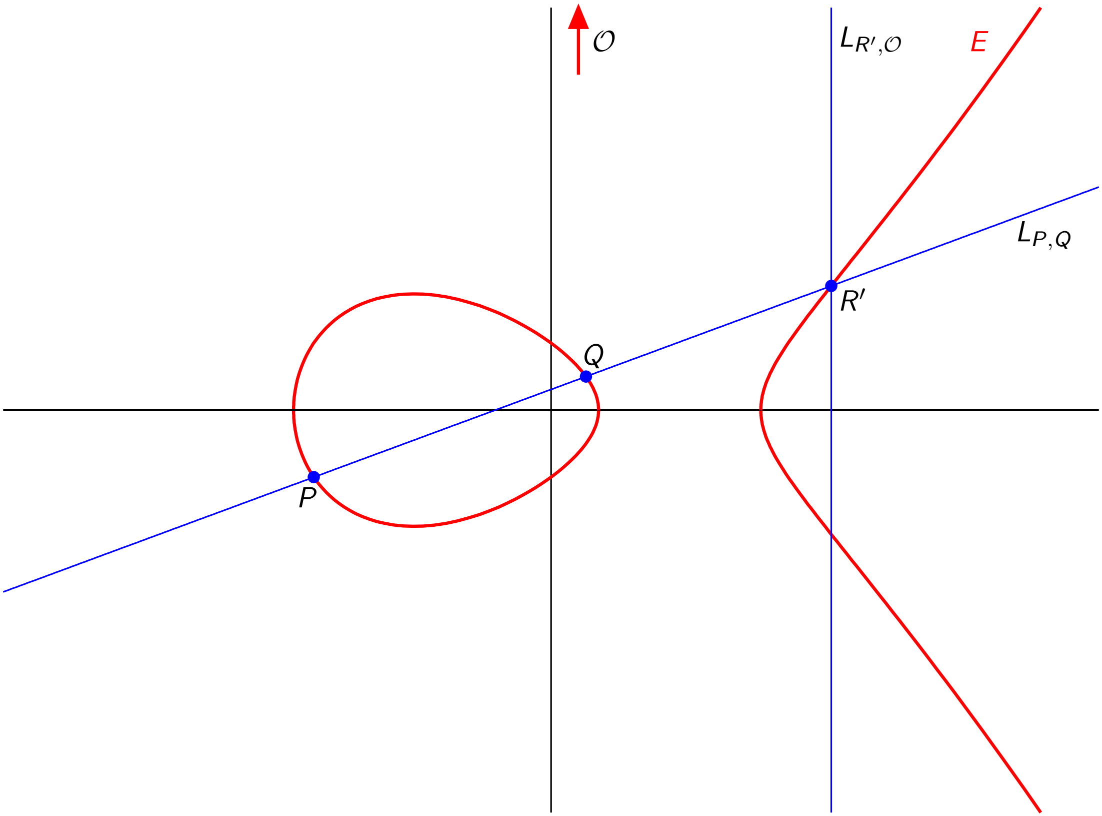

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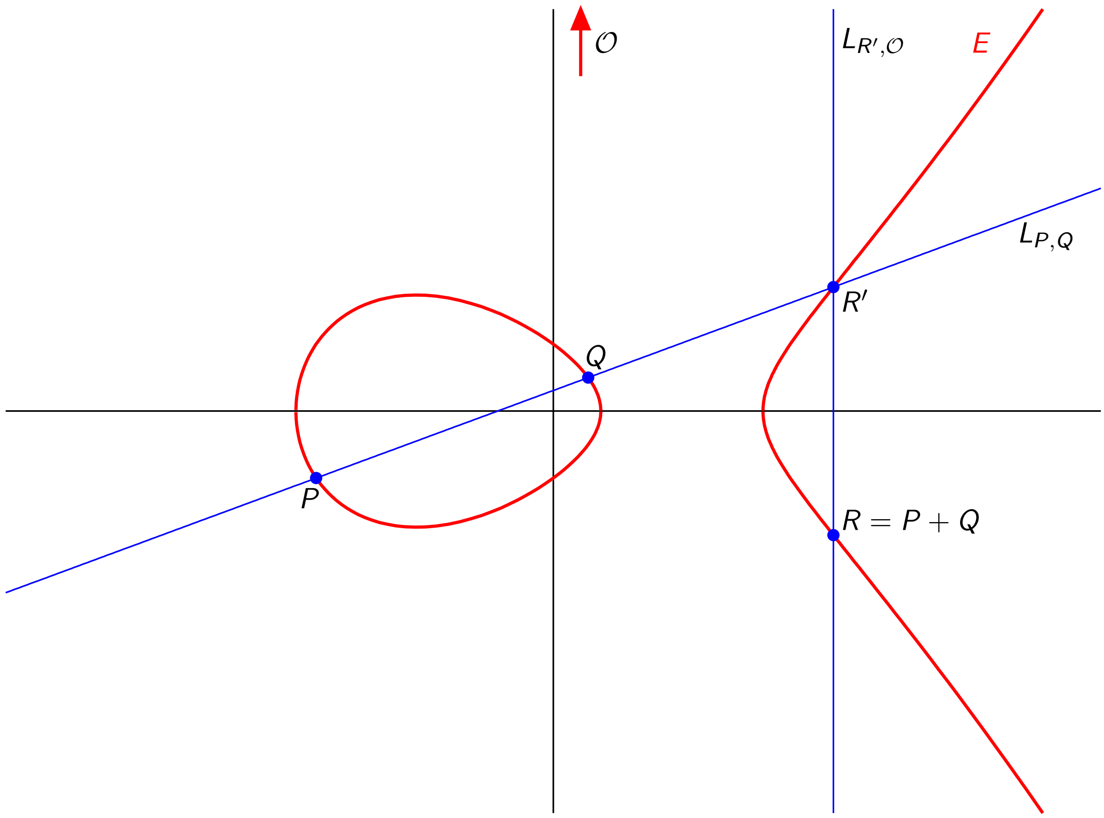

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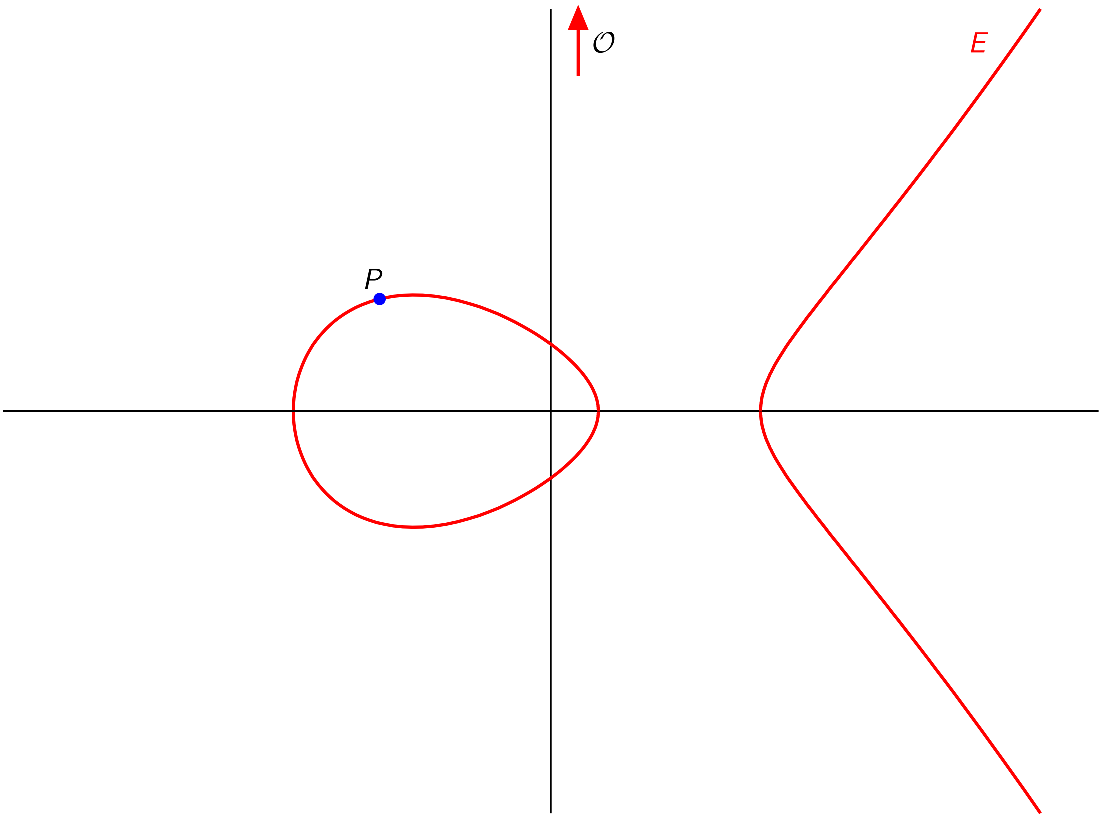

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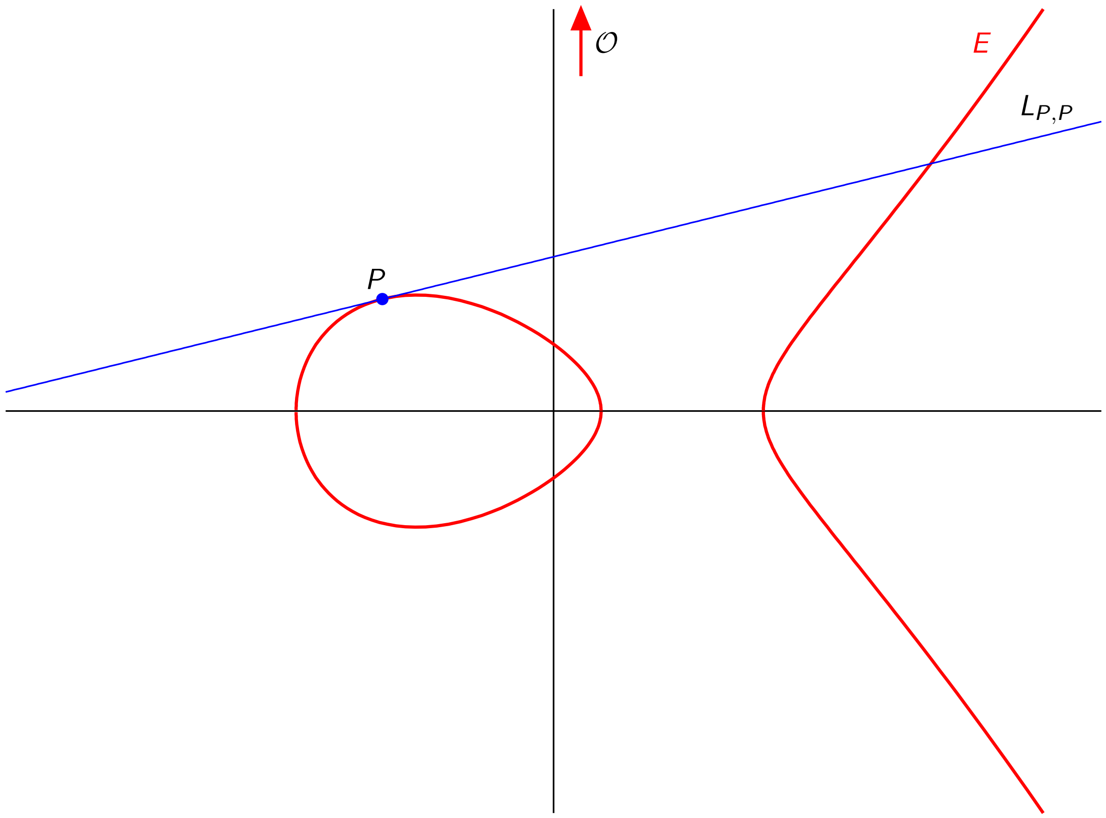

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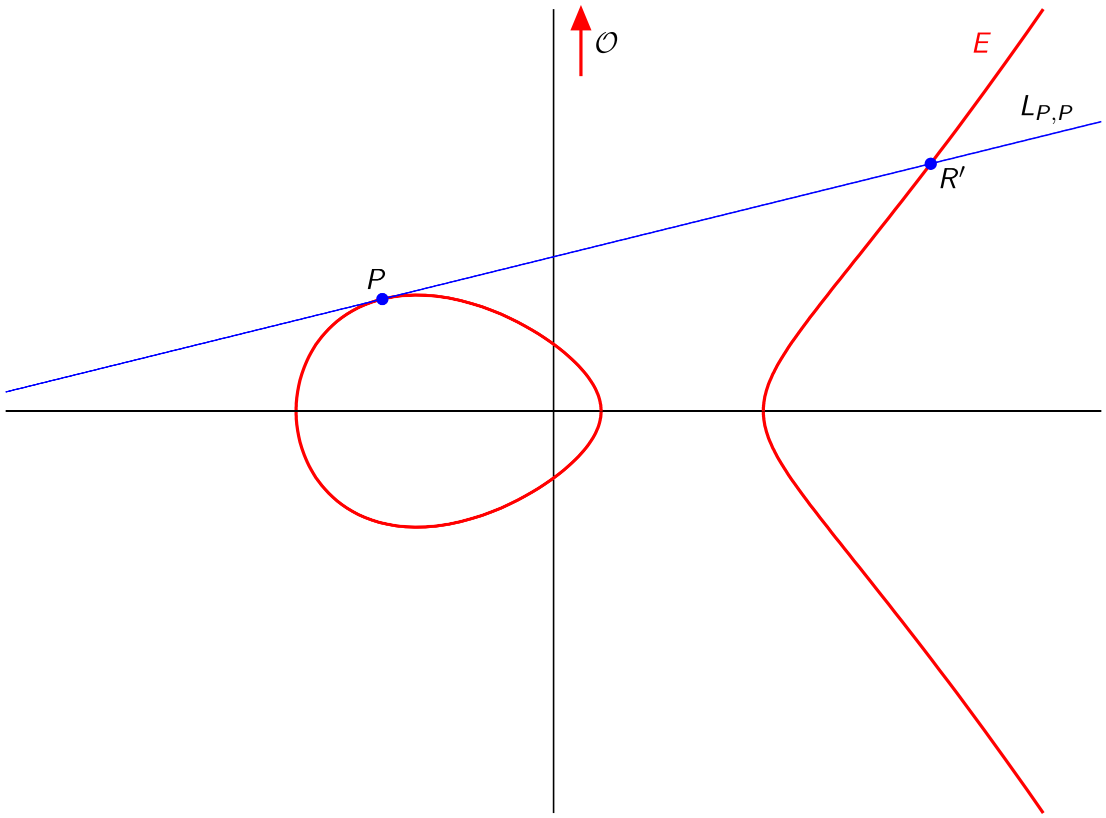

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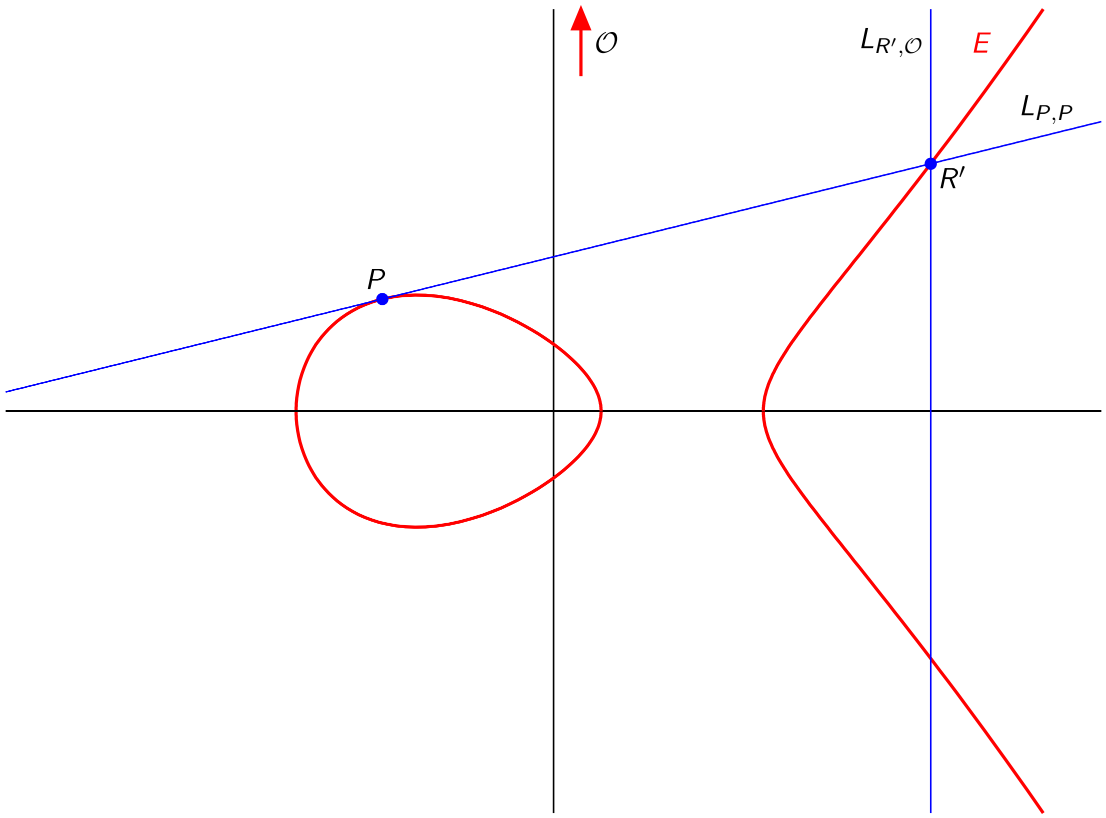

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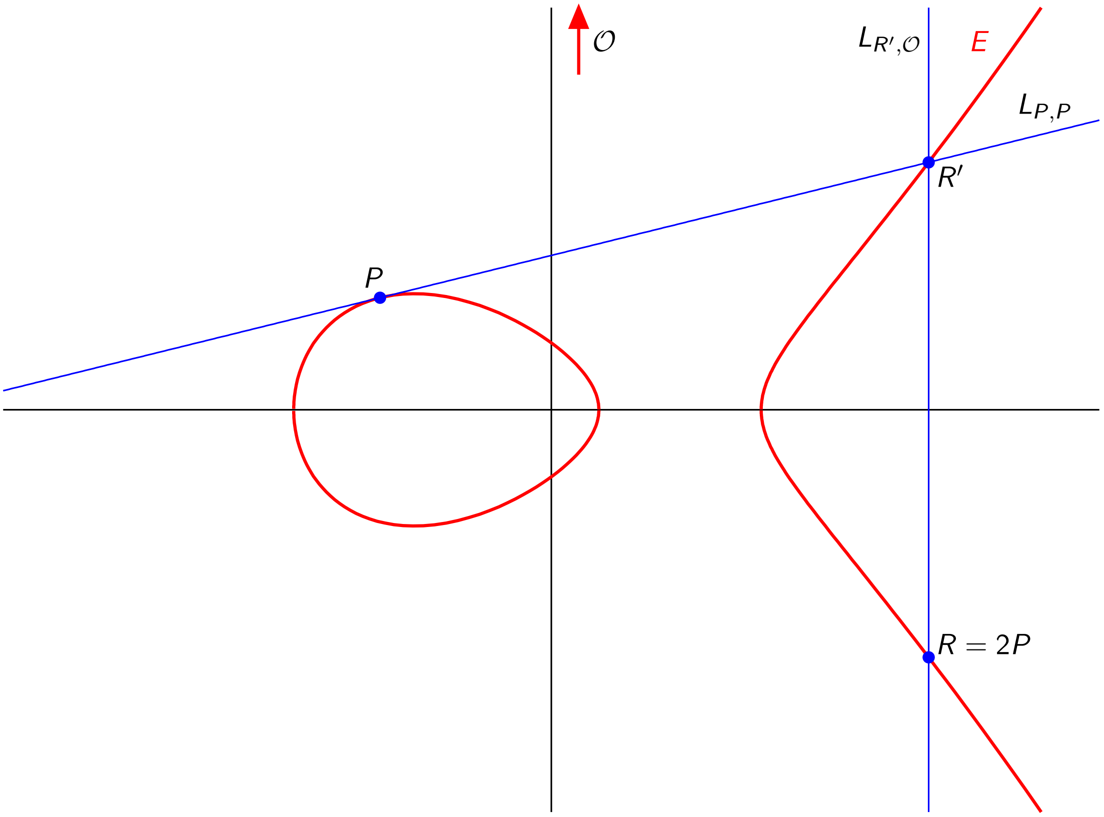

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$$x_R = \lambda^2 - x_P - x_Q \qquad \text{and} \qquad y_R = \lambda(x_P - x_R) - y_P$$

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$$\lambda = \begin{cases} \dfrac{y_Q - y_P}{x_Q - x_P} & \text{if } P \neq Q \text{ (addition), or} \\ \dfrac{3x_P^2 + A}{2y_P} & \text{if } P = Q \text{ (doubling)} \end{cases}$$

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  $\Rightarrow$ field inversion is expensive!

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- Explicit-Formula Database (by Bernstein and Lange):

`http://hyperelliptic.org/EFD/`

Outline

- Some encryption mechanisms
- Elliptic curve cryptography
- Scalar multiplication
- Elliptic curve arithmetic
- Finite field arithmetic

Implementing finite field arithmetic

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  $\Rightarrow$ elements of $\mathbb{F}_q$ represented using several words

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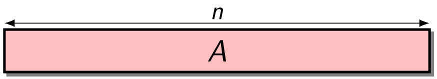

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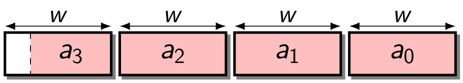
  - split A into $k = \lceil n/w \rceil$ w-bit words (or limbs), a_{k-1}, ..., a_1, a_0:

$$A = a_{k-1}2^{(k-1)w} + \cdots + a_1 2^w + a_0$$



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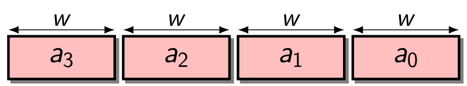

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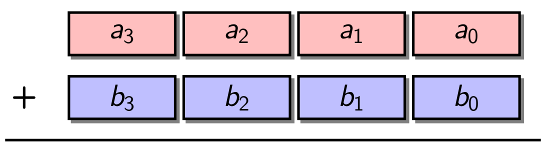

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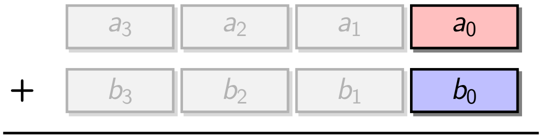

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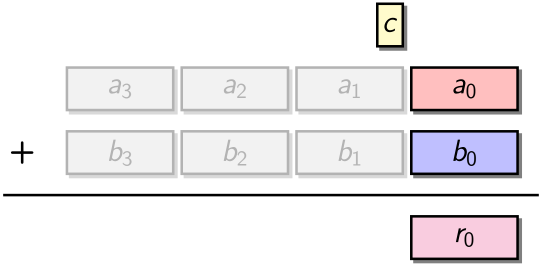

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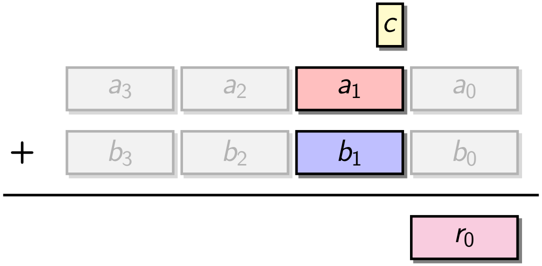

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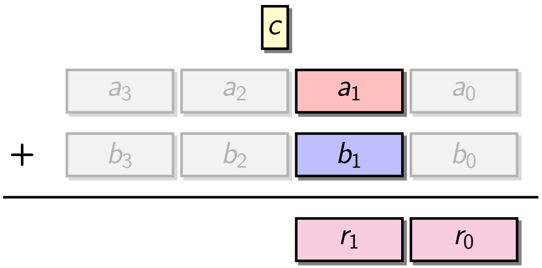

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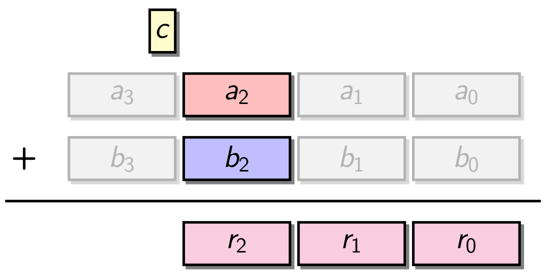

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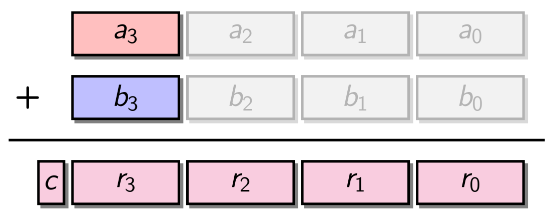

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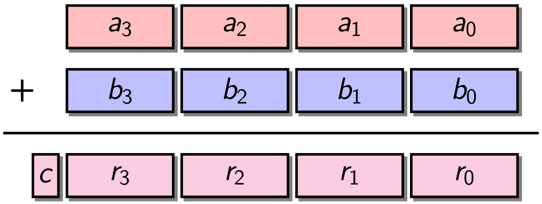

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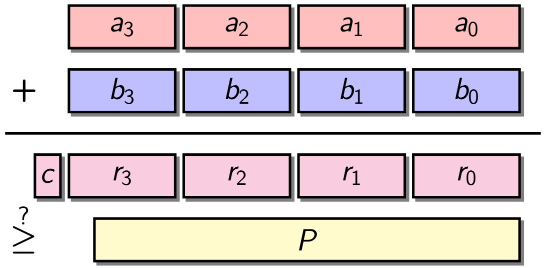

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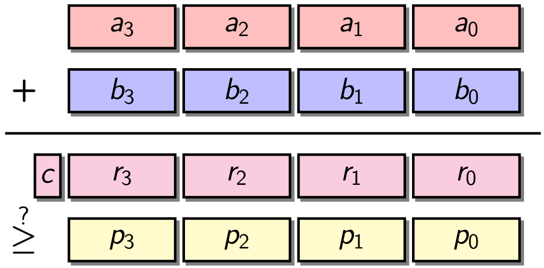

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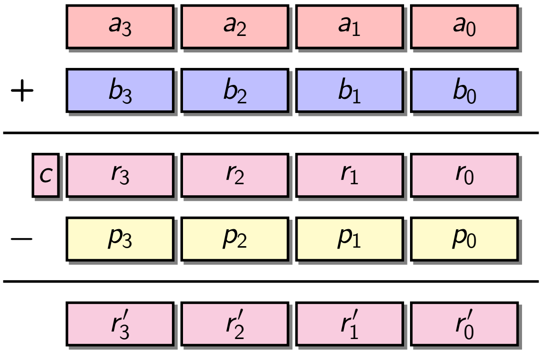

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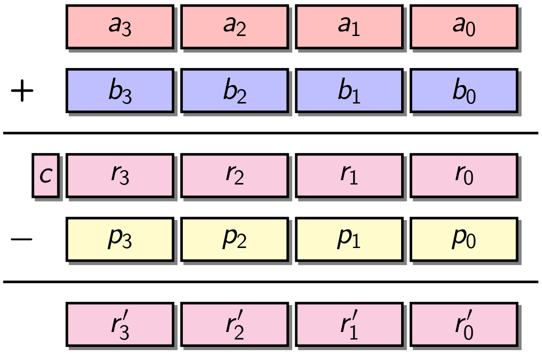

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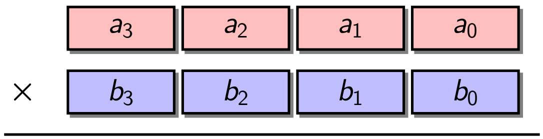

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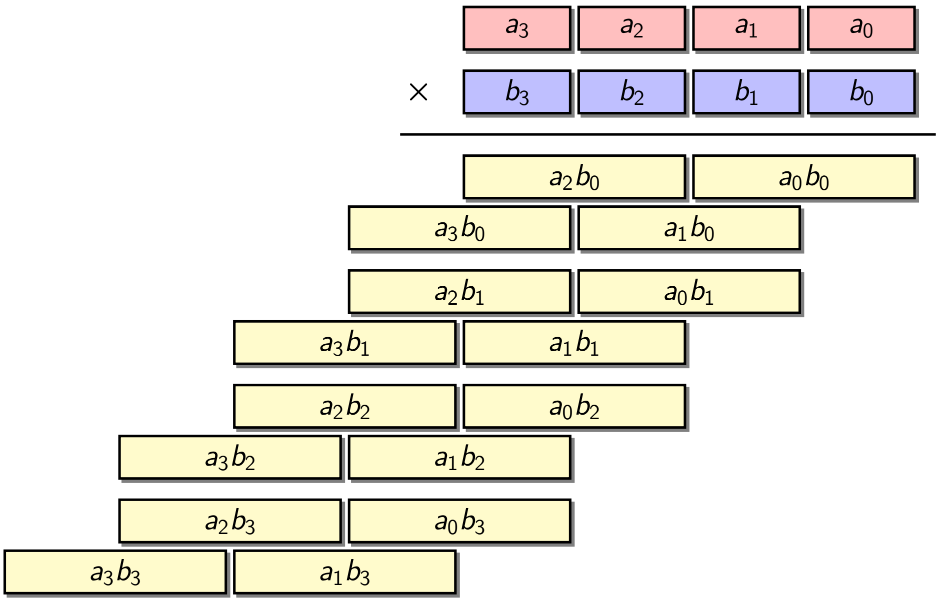

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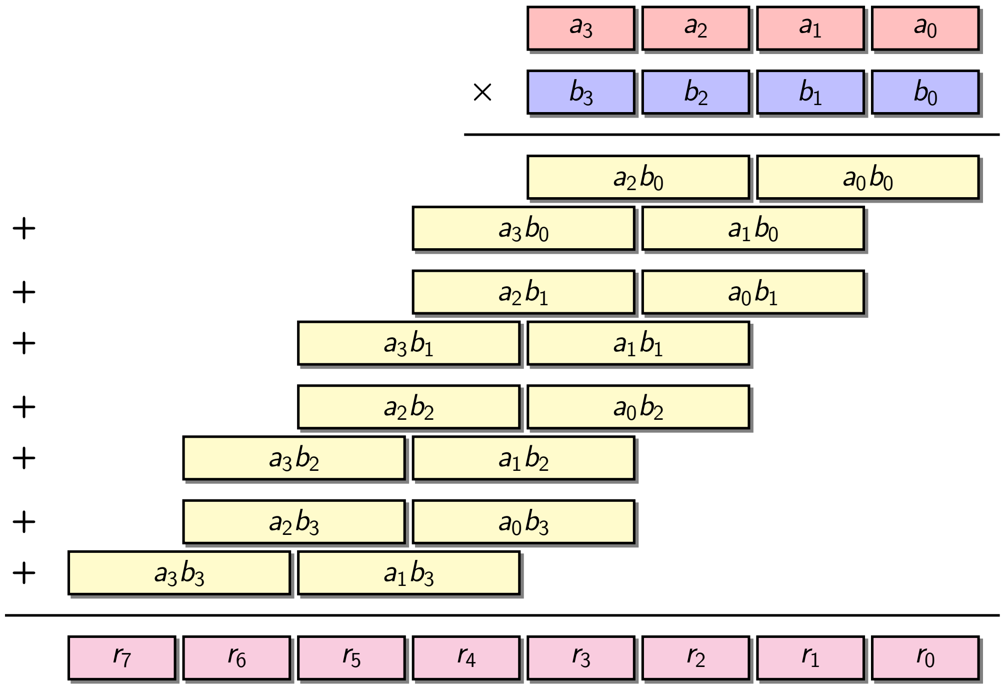

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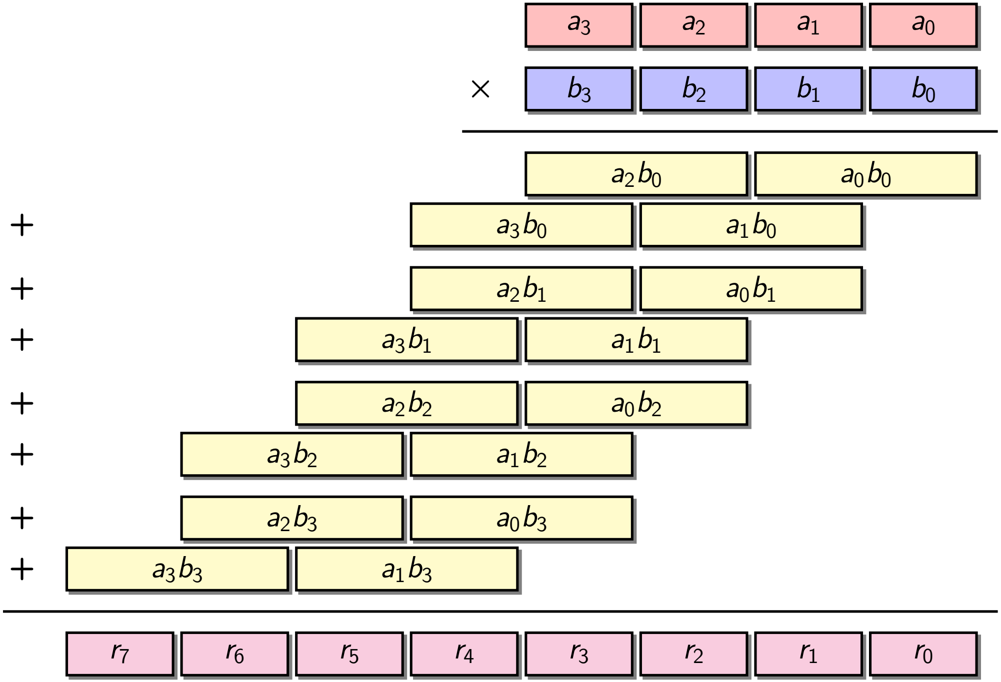

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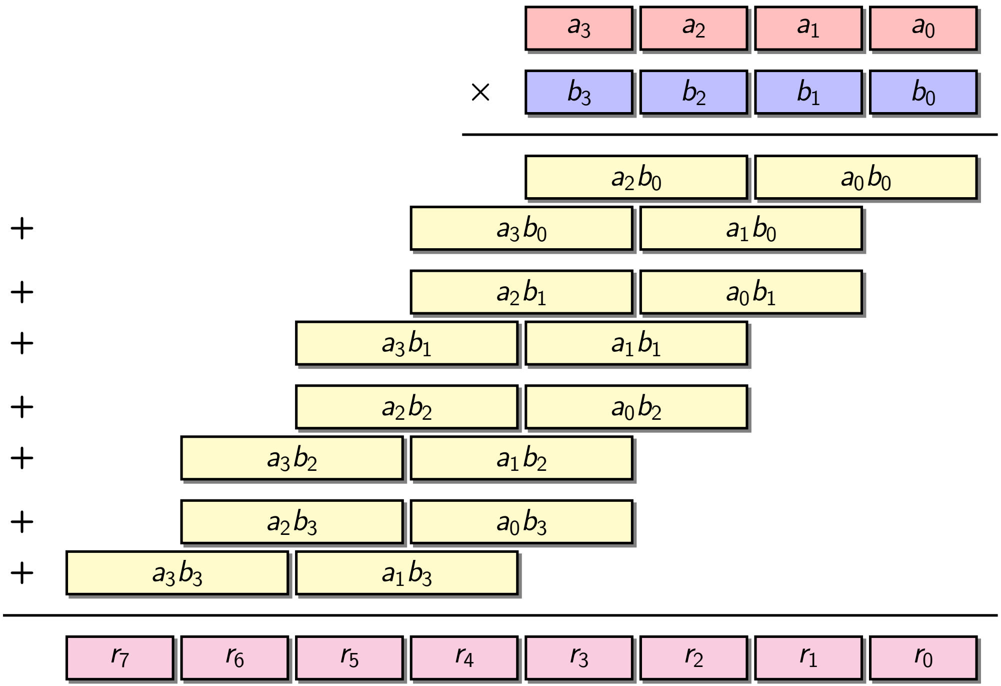

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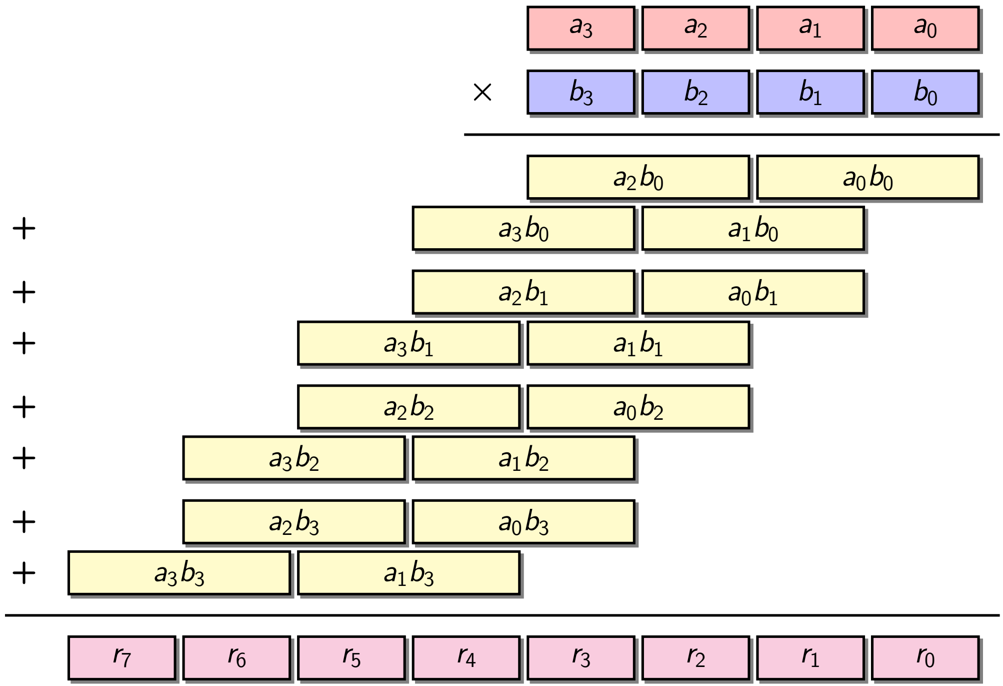

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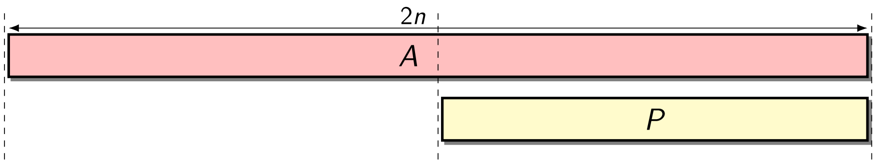

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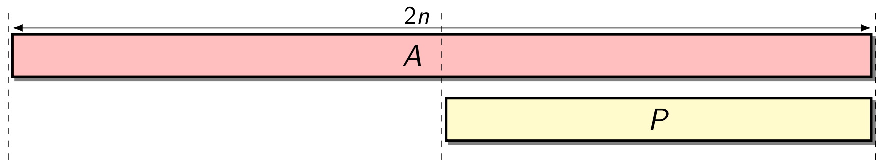

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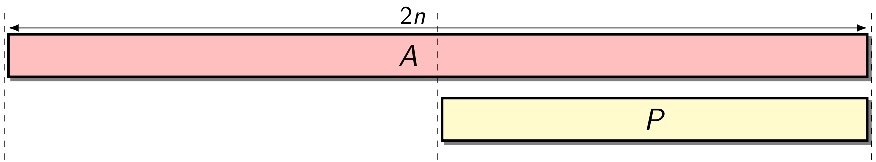

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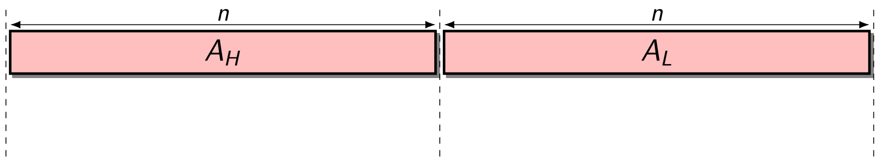

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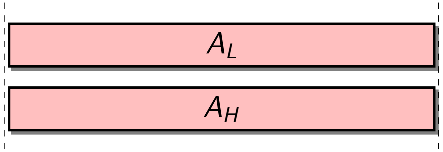

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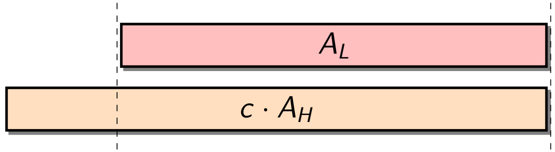

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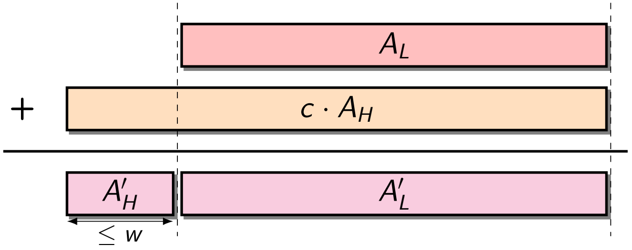

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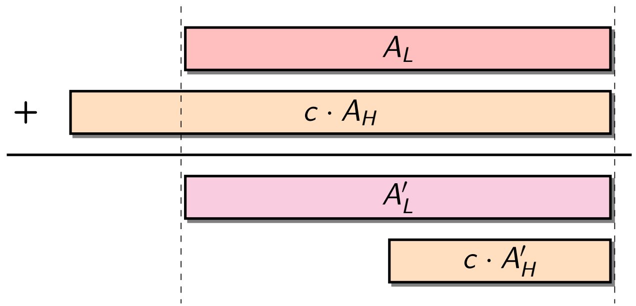

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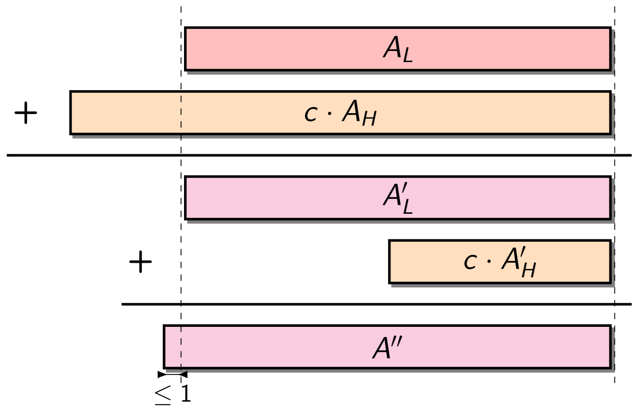

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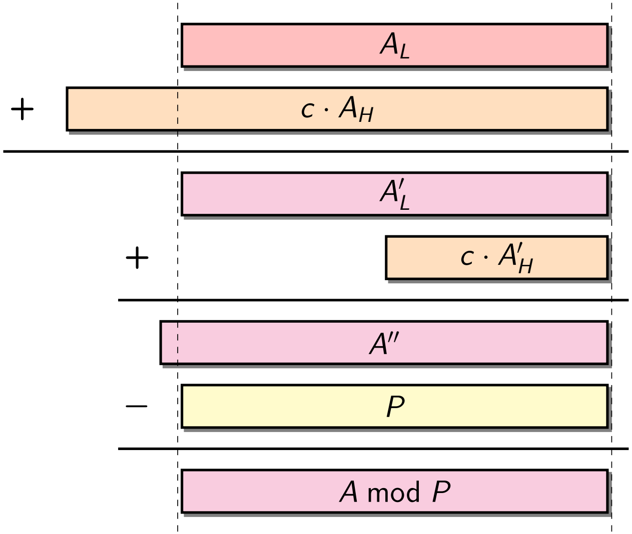

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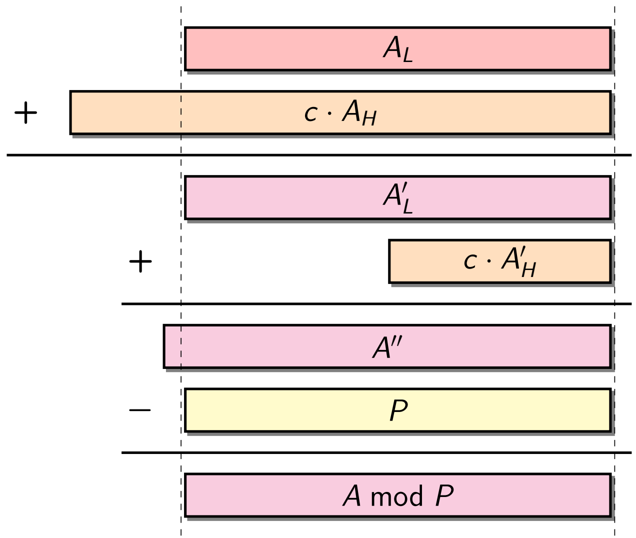
- Examples: $P = 2^{255} - 19$ (Curve25519) or $P = 2^{448} - 2^{224} - 1$ (Ed448-Goldilocks)



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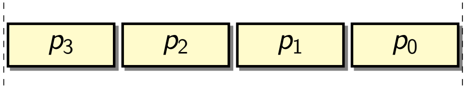

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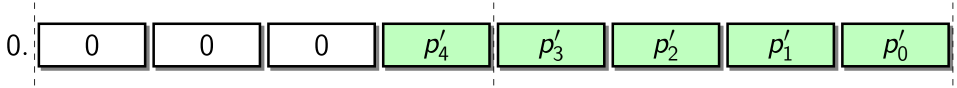

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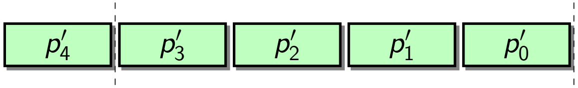

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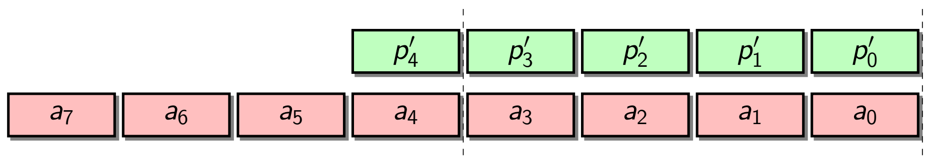

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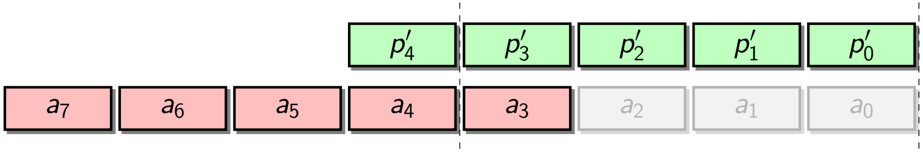

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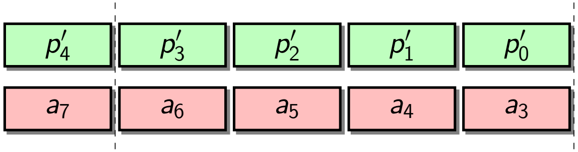

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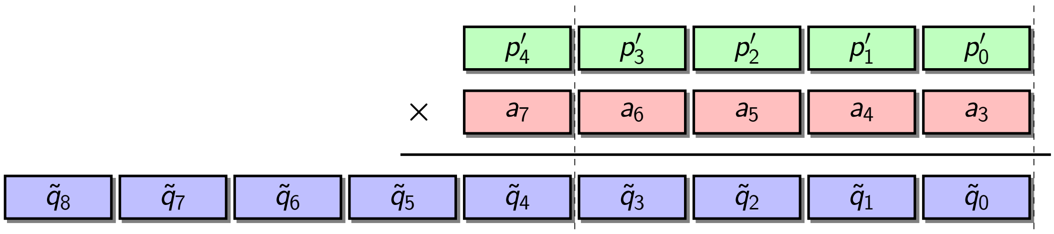

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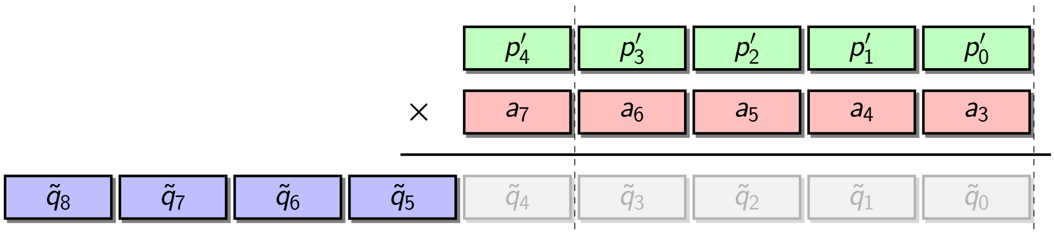

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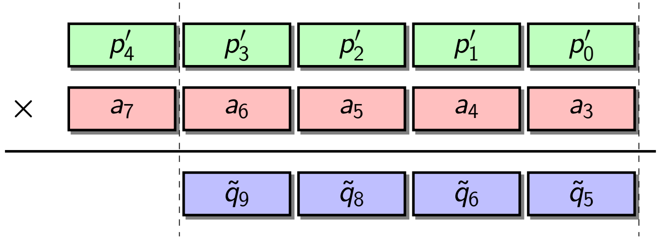

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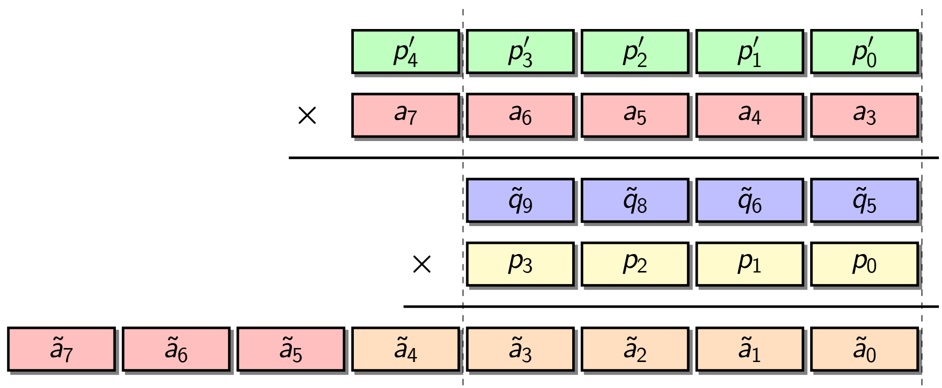

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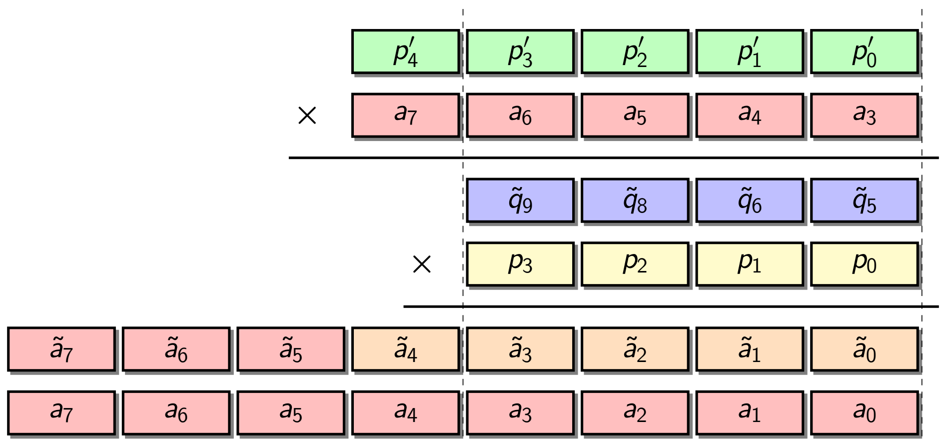

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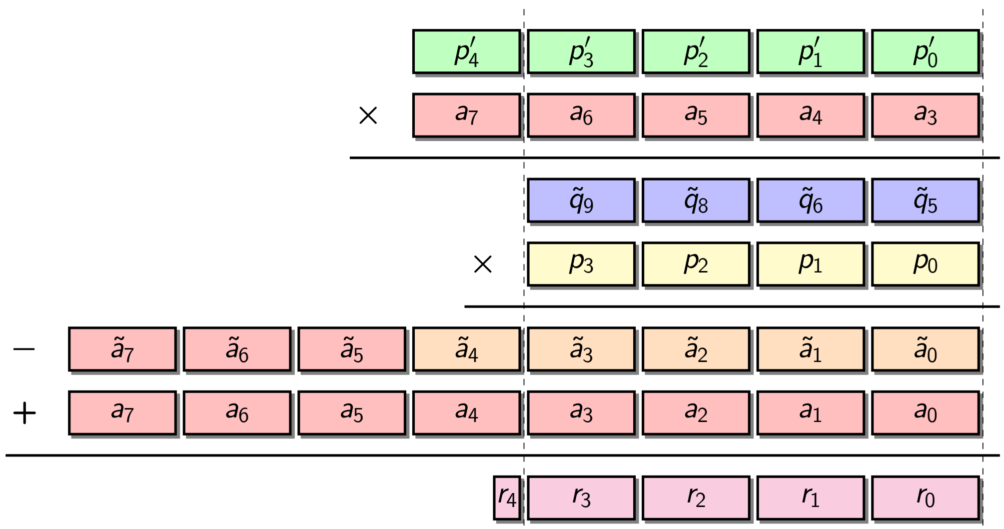

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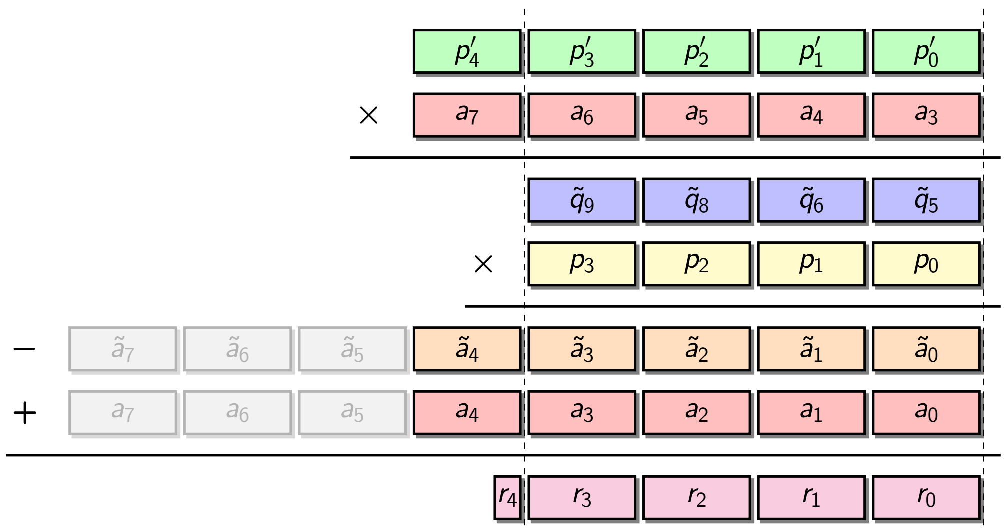

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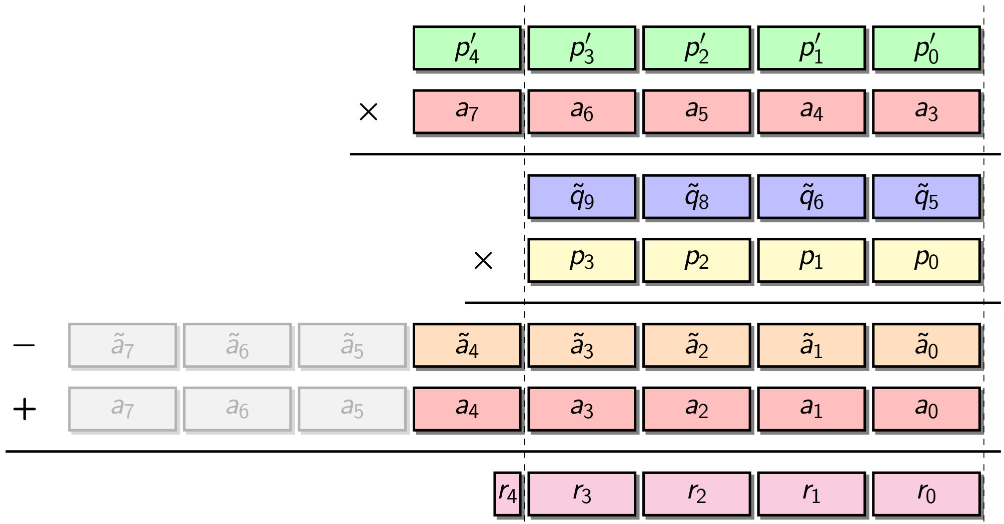

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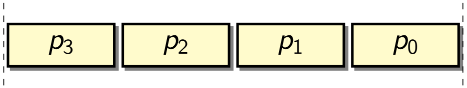

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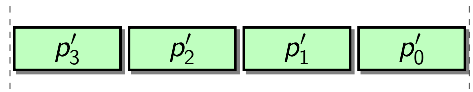

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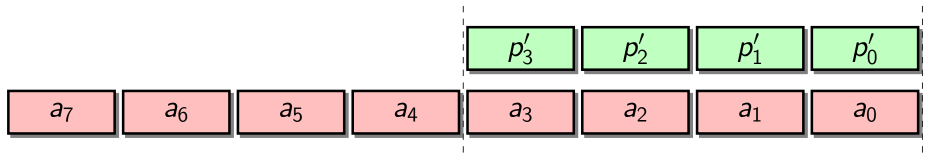

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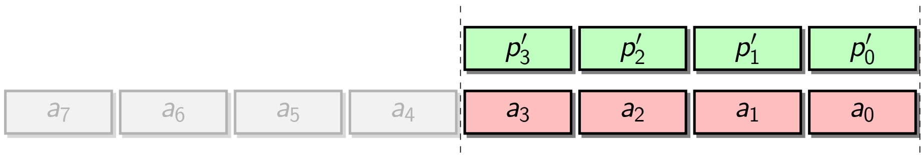

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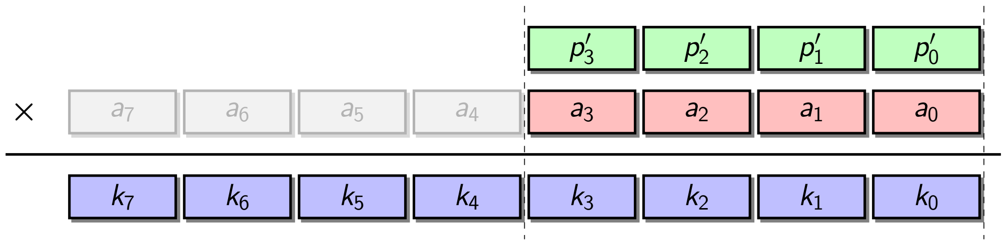

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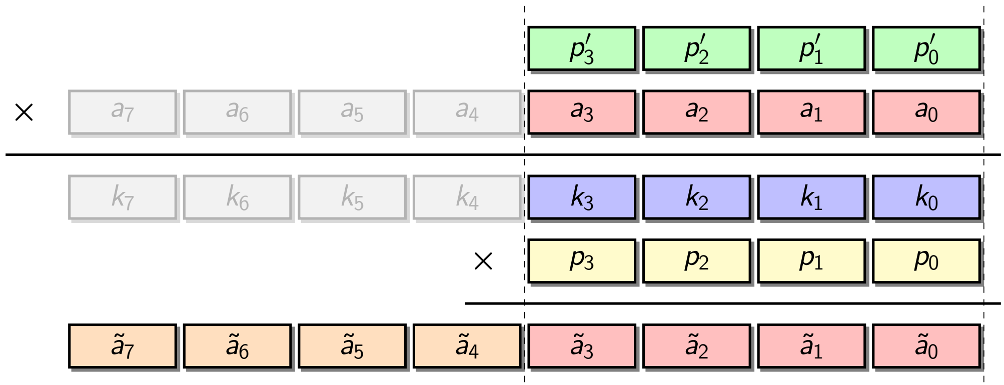

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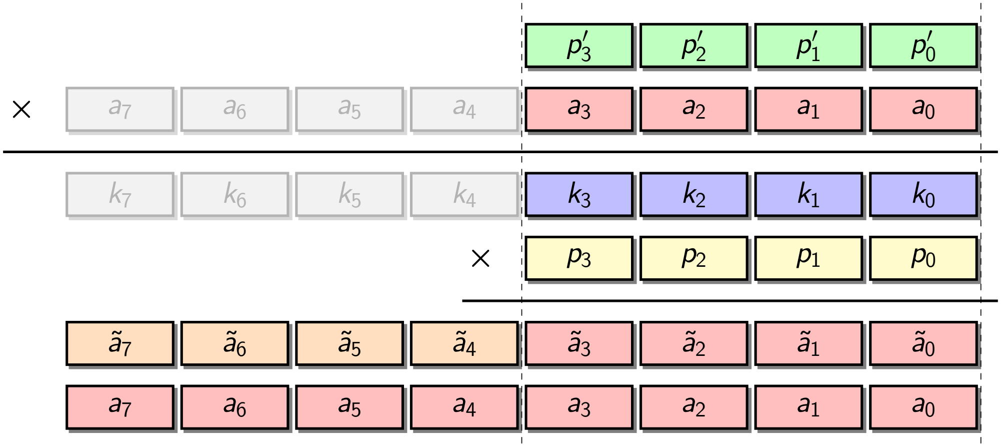

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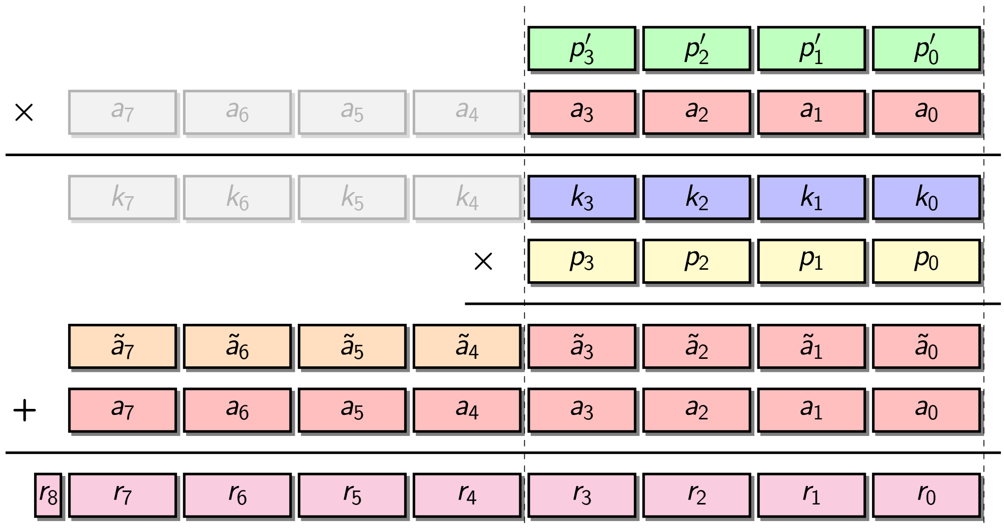

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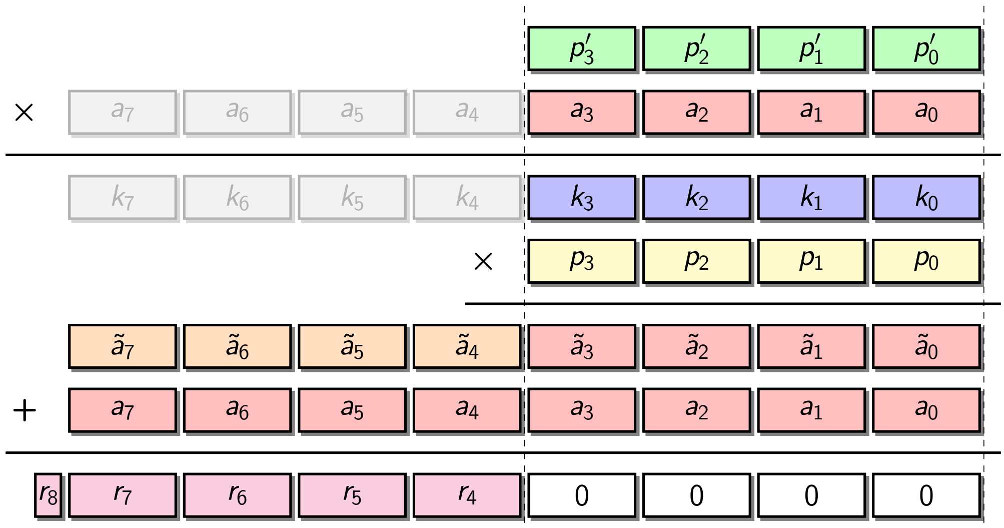

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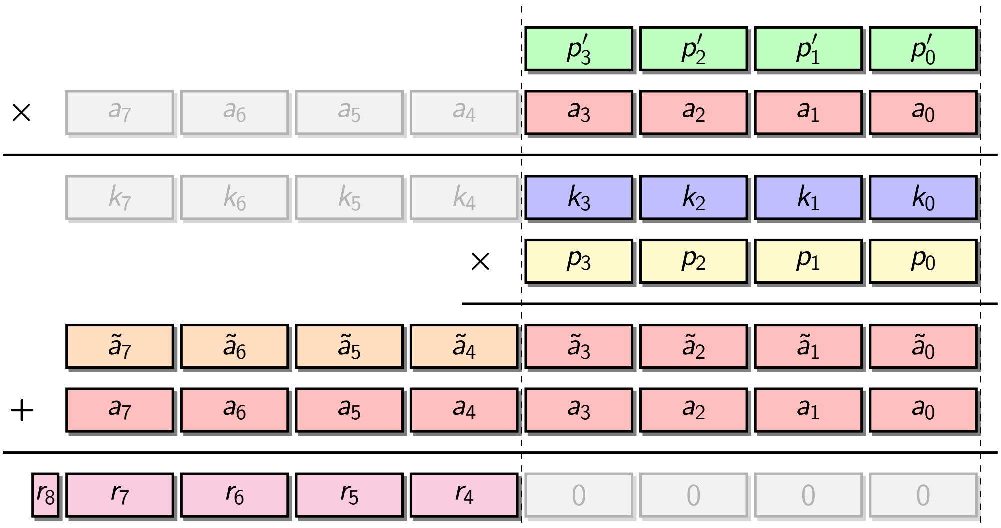

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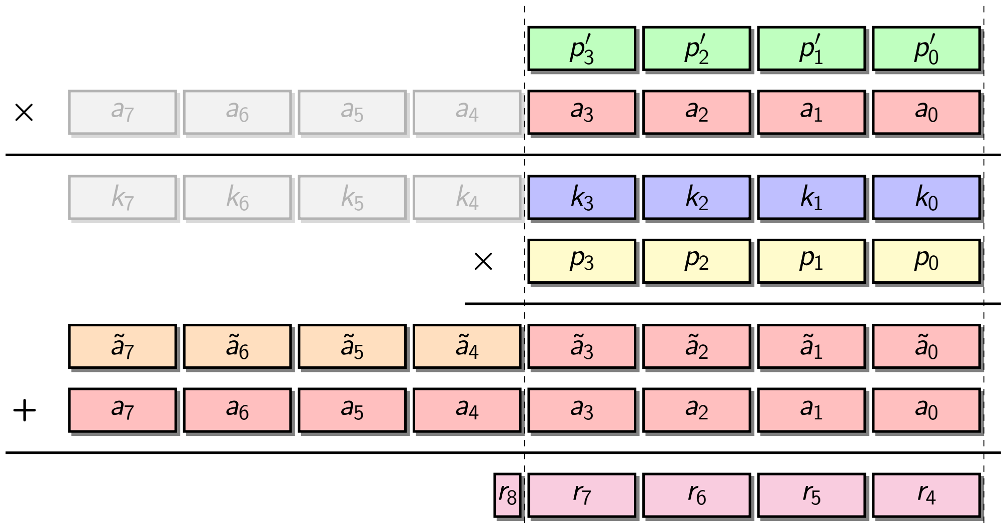

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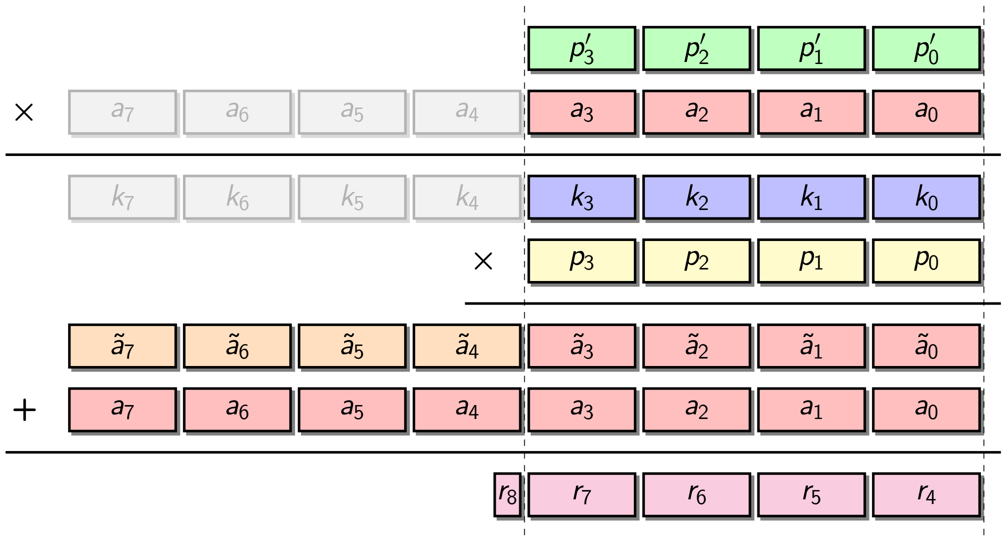

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- REDC can be computed iteratively (one word at a time) and interleaved with the computation of $\hat{X} \cdot \hat{Y}$

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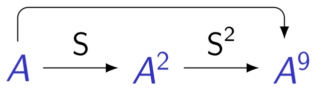

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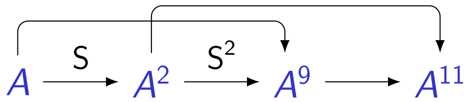

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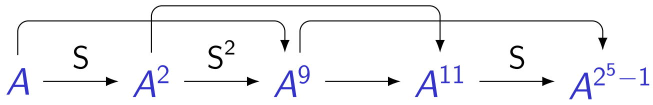

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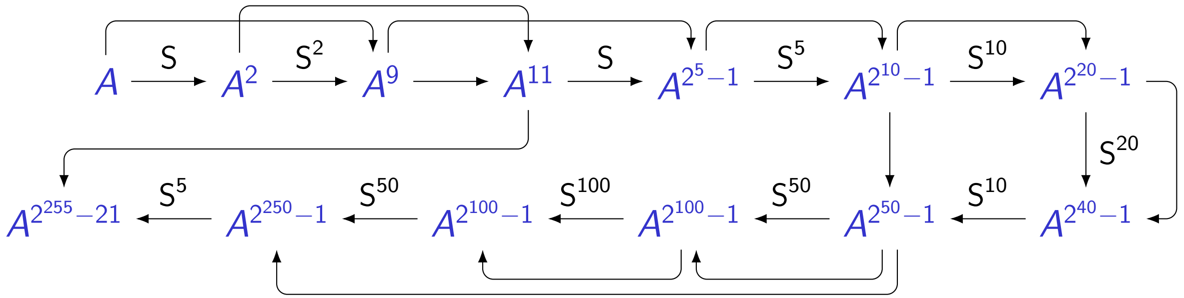

The Residue Number System (RNS)

- Let $\mathcal{B} = (m_1, \ldots, m_k)$ a tuple of k pairwise coprime integers
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$$m_i = 2^w - c_i, \text{ with small } c_i$$

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- If $M > P$, we can represent elements of $\mathbb{F}_P$ in RNS

RNS arithmetic

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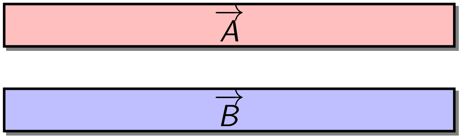

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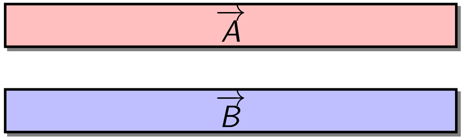

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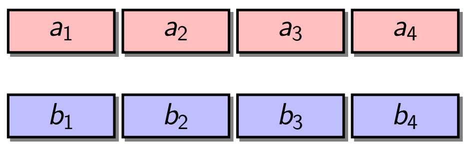

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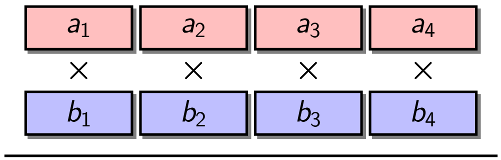

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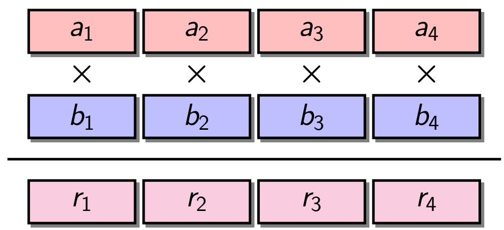

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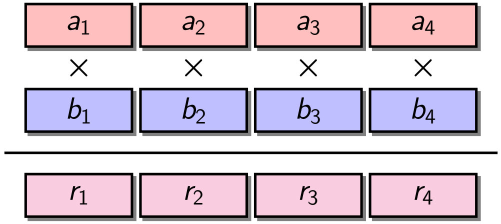

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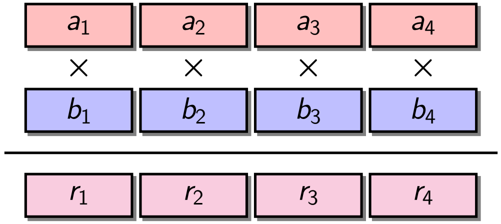

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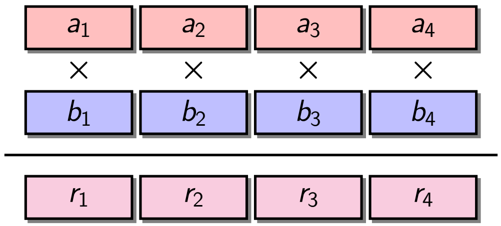

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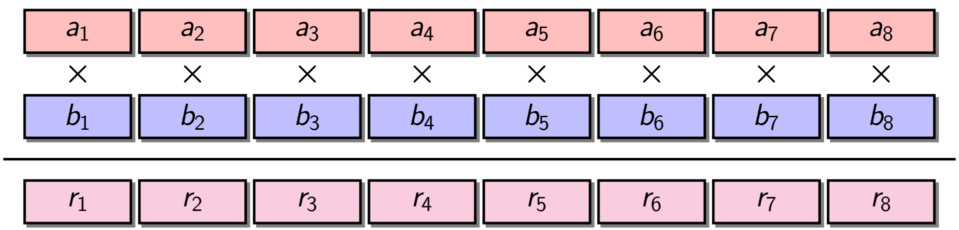

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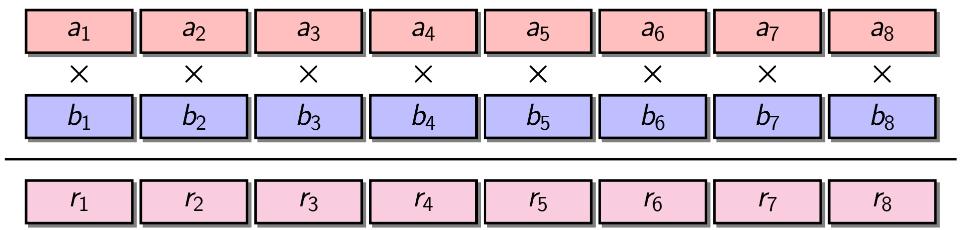

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  - RNS modular reduction has quadratic complexity $O(k^2)$

RNS Montgomery reduction

- Requires two RNS bases $\mathcal{B}_\alpha = (m_{\alpha,1}, \ldots, m_{\alpha,k})$ and $\mathcal{B}_\beta = (m_{\beta,1}, \ldots, m_{\beta,k})$ such that $M_\alpha > P$, $M_\beta > P$, and $\gcd(M_\alpha, M_\beta) = 1$

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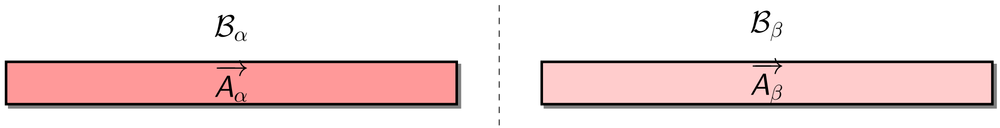

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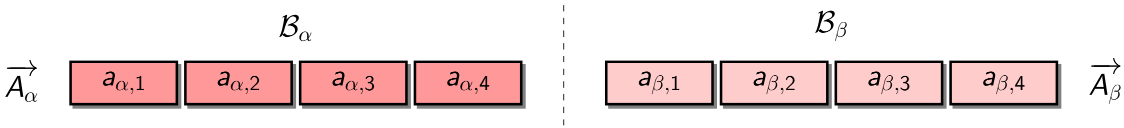

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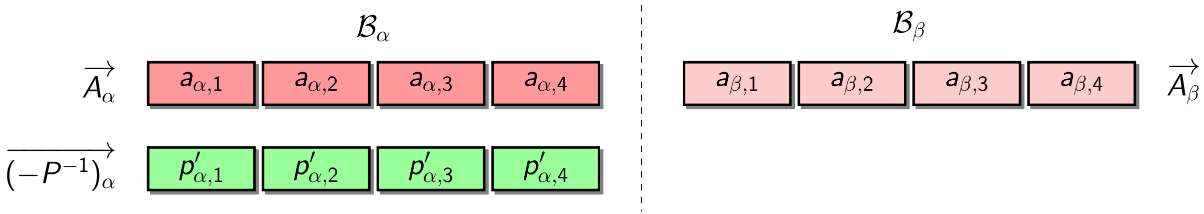

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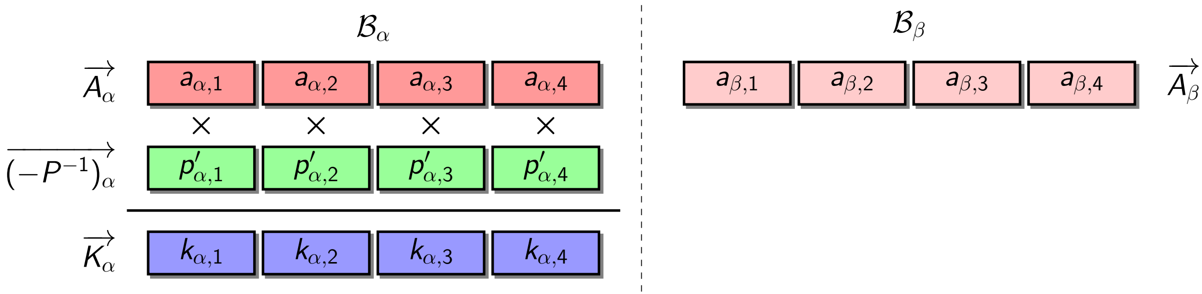

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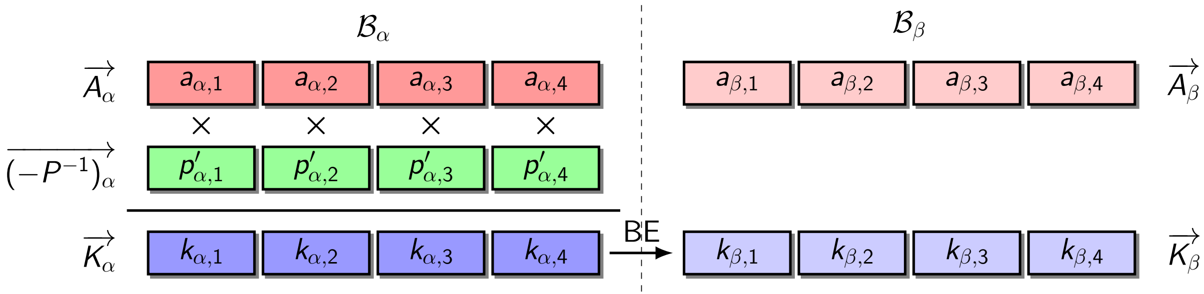

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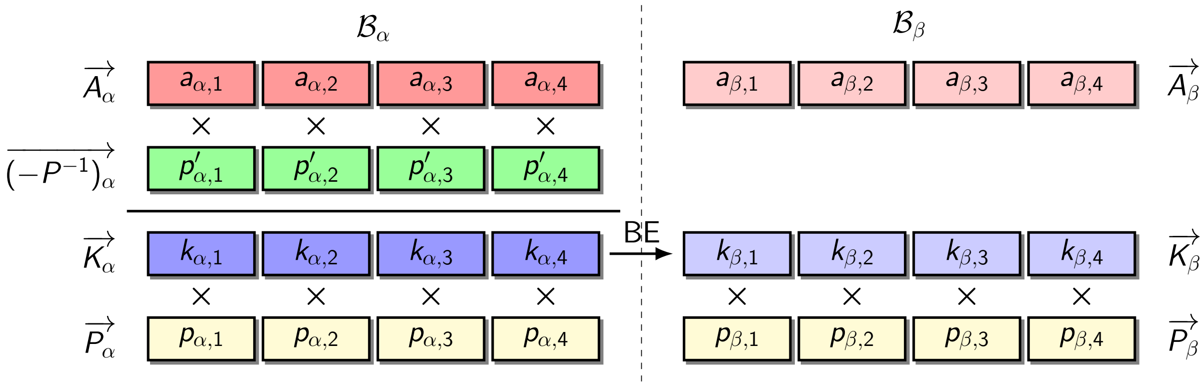

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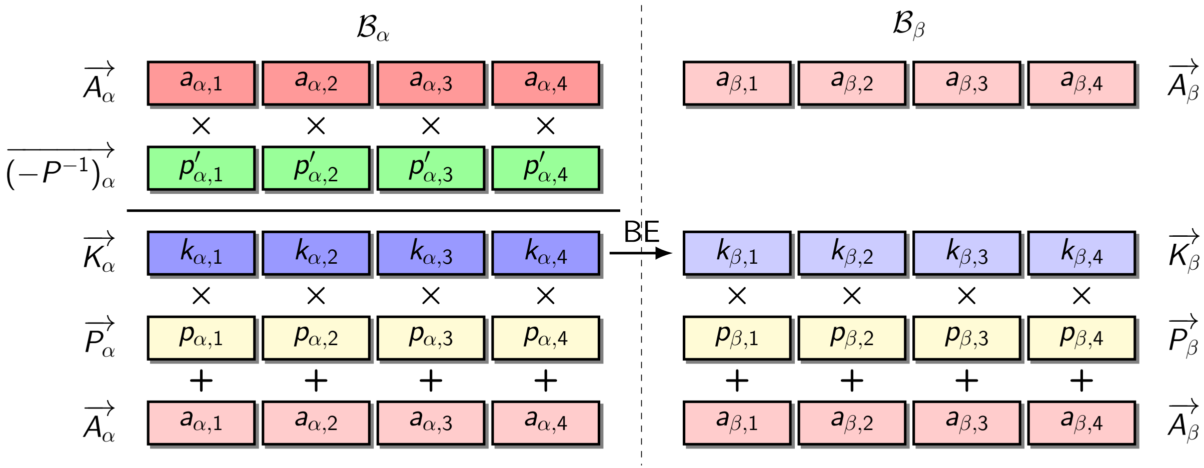

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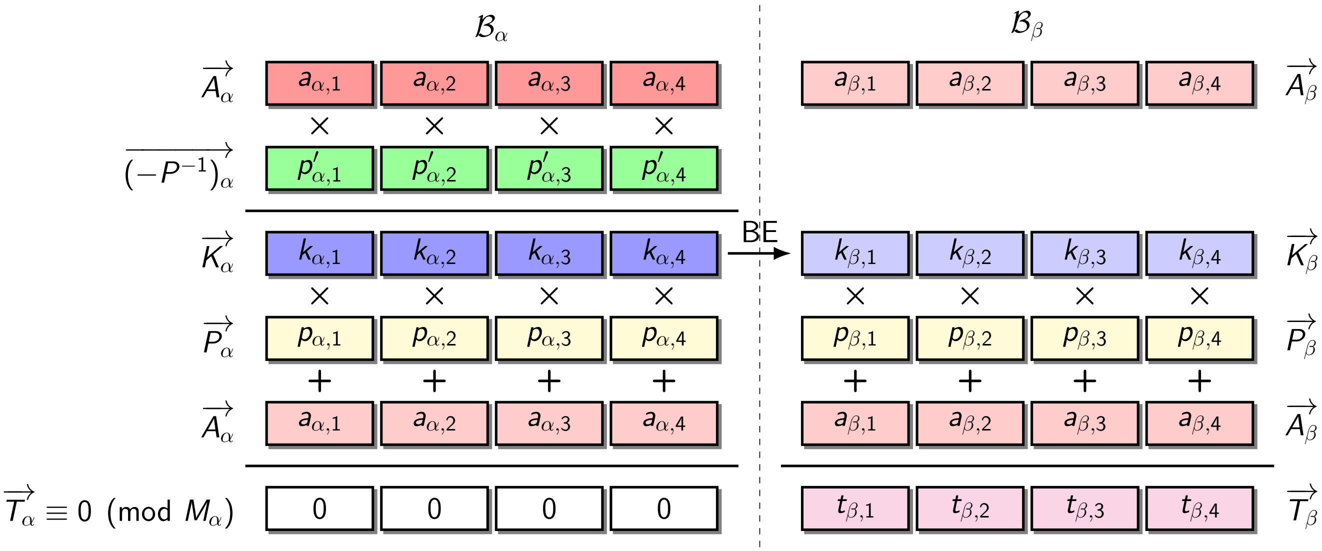
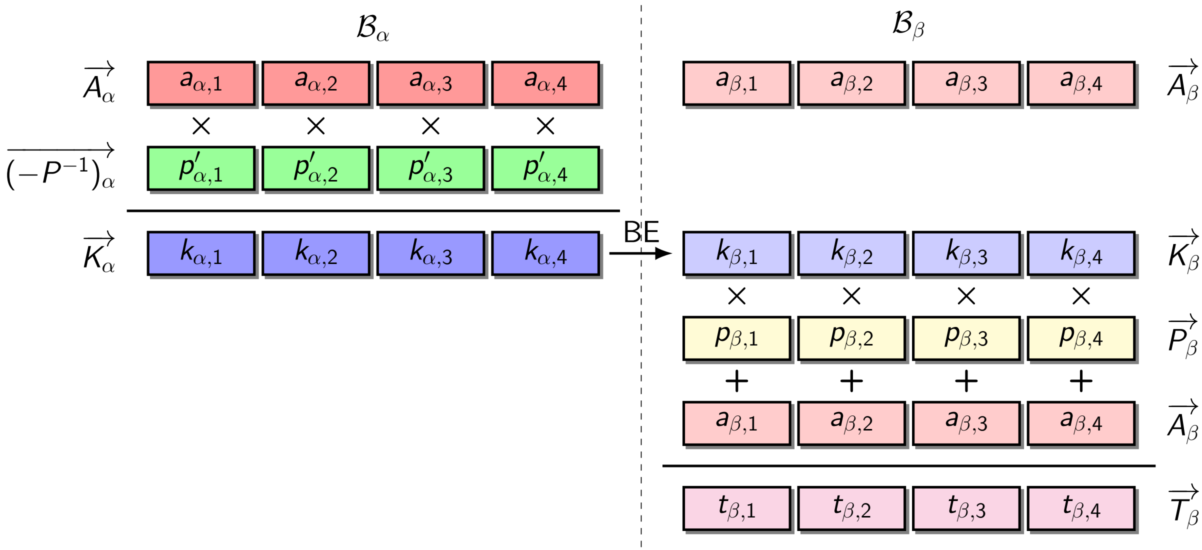
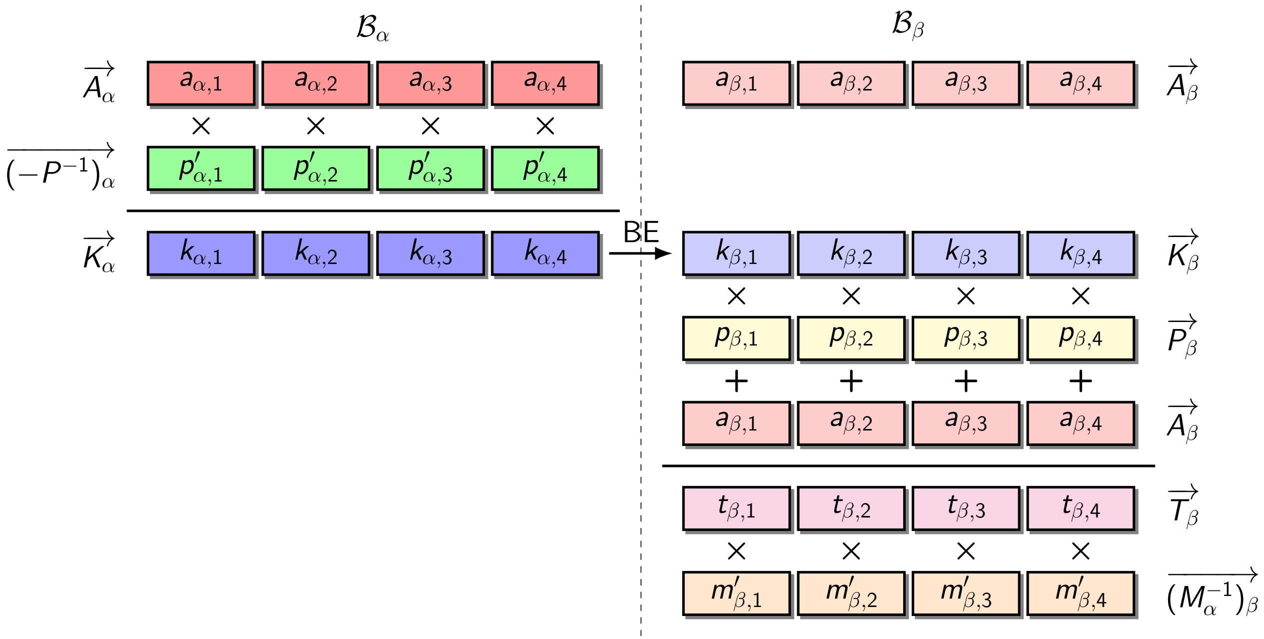


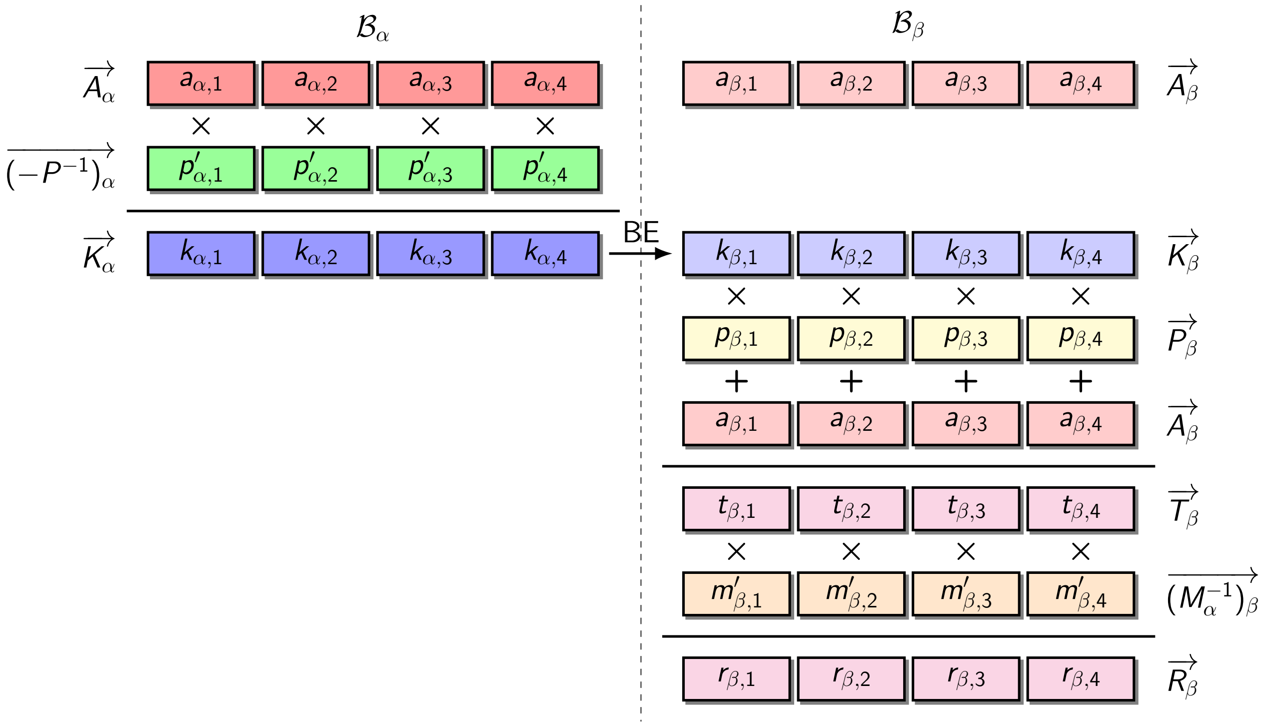
RNS Montgomery reduction



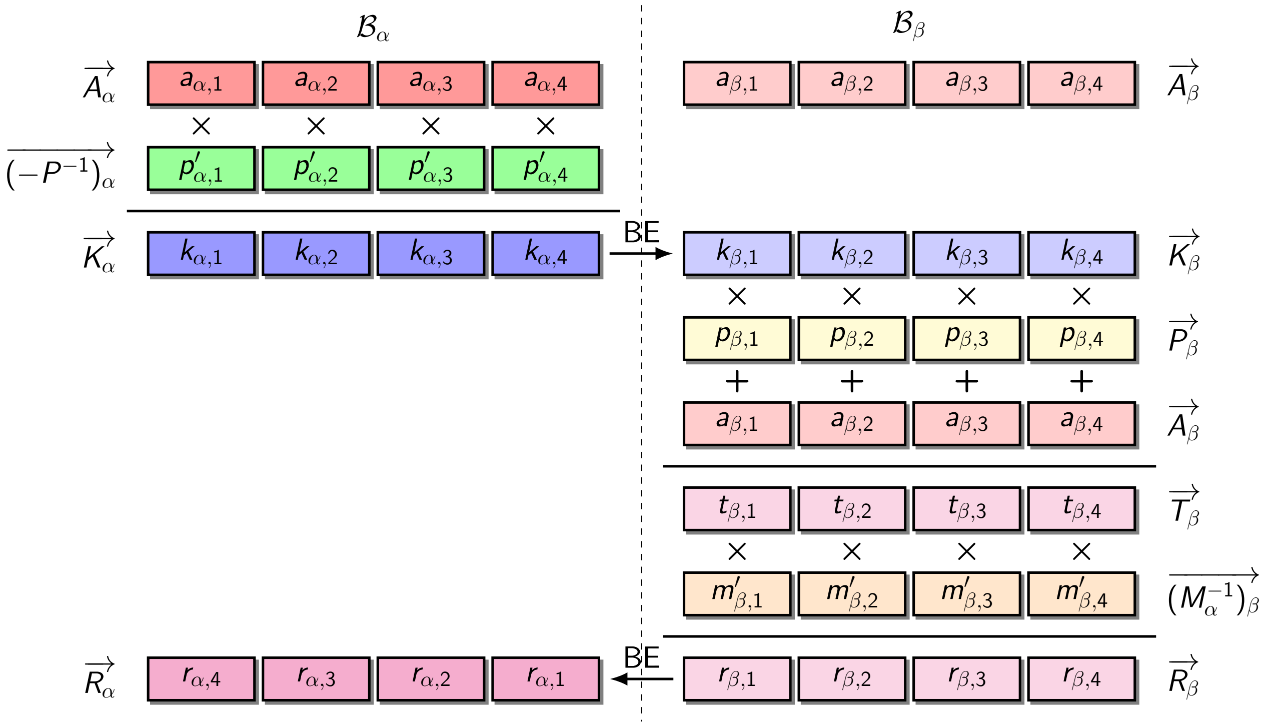
RNS Montgomery reduction



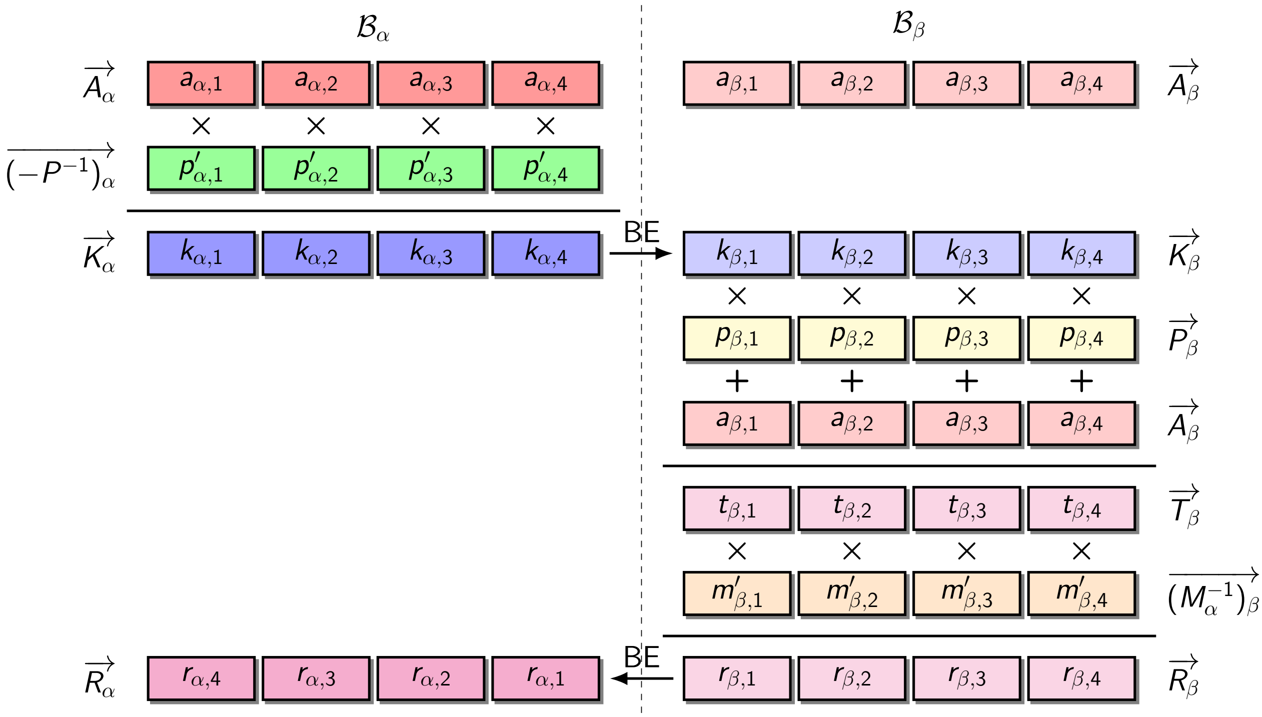
RNS Montgomery reduction



RNS Montgomery reduction

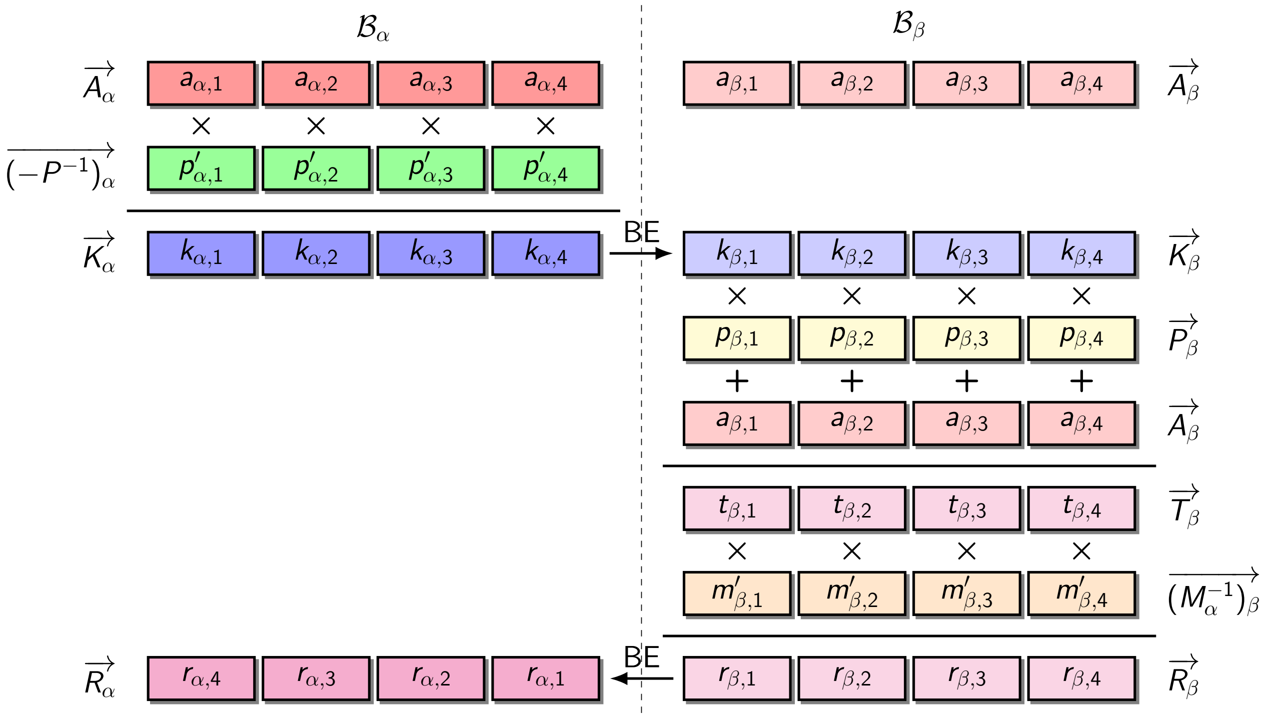


RNS Montgomery reduction



- Result is $(\overrightarrow{R_\alpha}, \overrightarrow{R_\beta}) \equiv (A \cdot M_\alpha^{-1}) \pmod{P}$

RNS Montgomery reduction



- Result is $(\overrightarrow{R_\alpha}, \overrightarrow{R_\beta}) \equiv (A \cdot M_\alpha^{-1}) \pmod{P}$
- See also the hybrid position–residues number system [Bigou & Tisserand, 2016]

Un peu de publicité éhontée...

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https://jc2-2017.inria.fr/

À très bientôt dans les Vosges !